Computing with Large Populations

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AlgoGT
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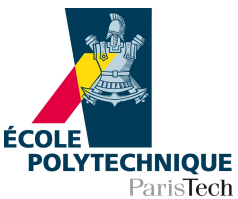

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Inspiration: Classical Population Protocols

Large Population Protocols

Discussion and Conclusion

Inspiration: (classical) Population Protocols

Introduced by [Angluin,Aspnes,Diamadi,Fischer,Peralta 2004] in the context of distributed systems.

A model of anonymous (sensor) networks with absolutely minimal assumptions about

- sophistication of mobile units: finite state machines
- infrastructure: none (not even unique ids)
- synchrony: totally asynchronous
- communication range: big enough that communication is occasionally possible between two agents.

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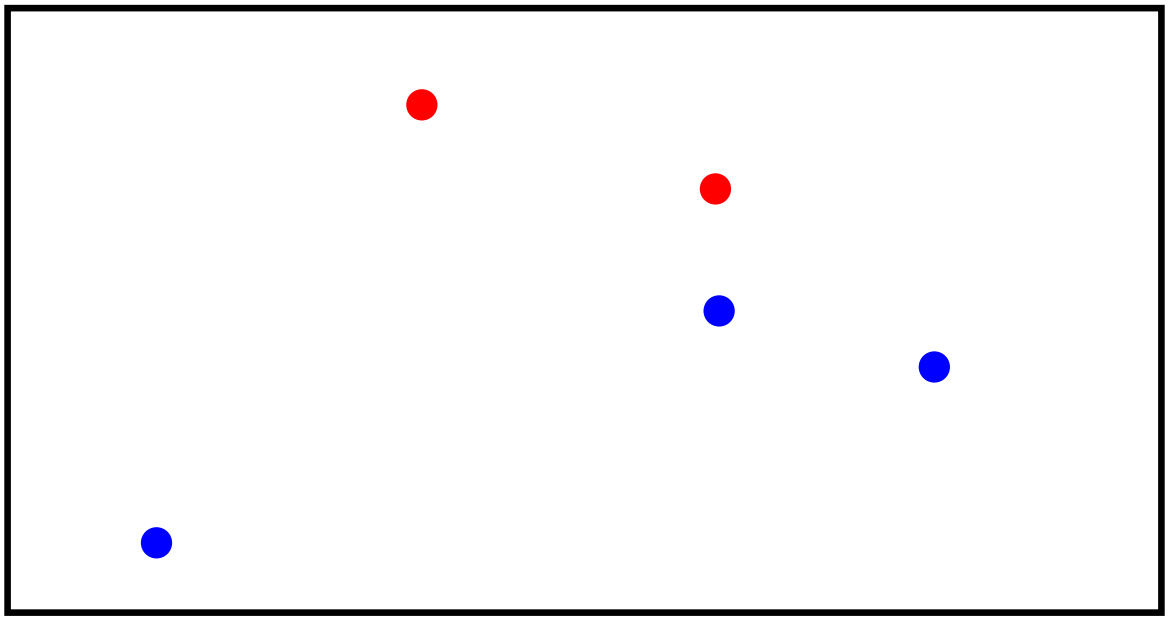

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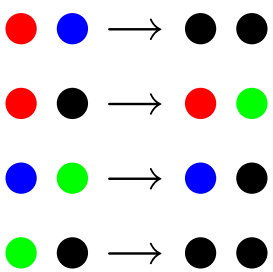

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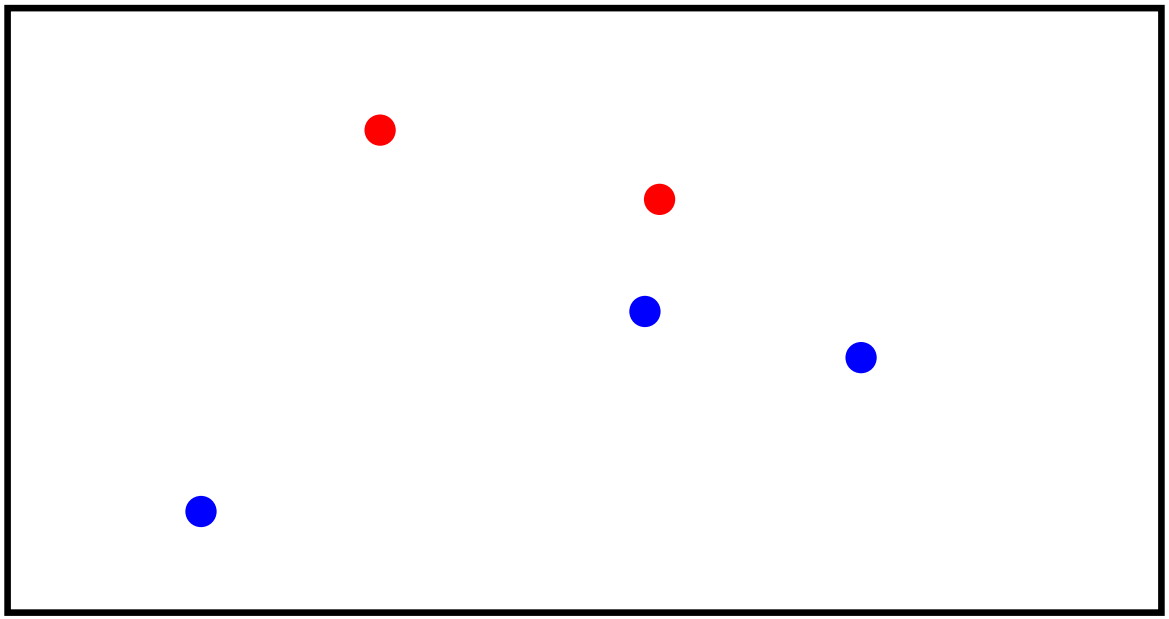

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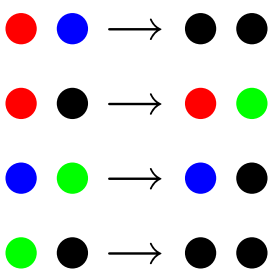

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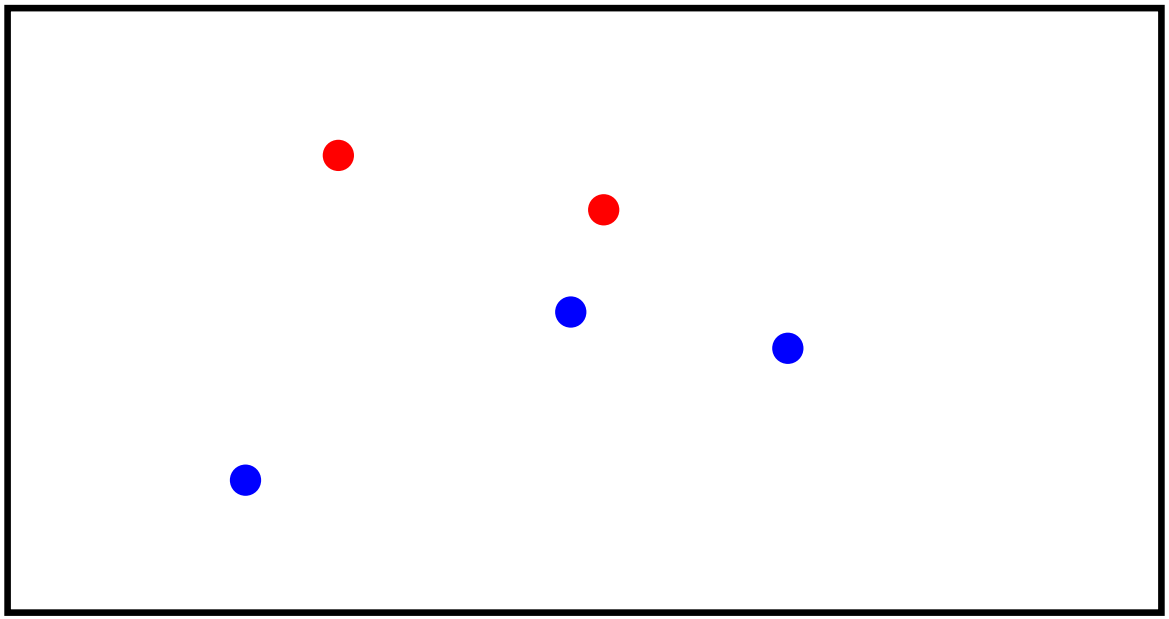

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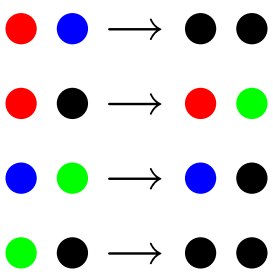

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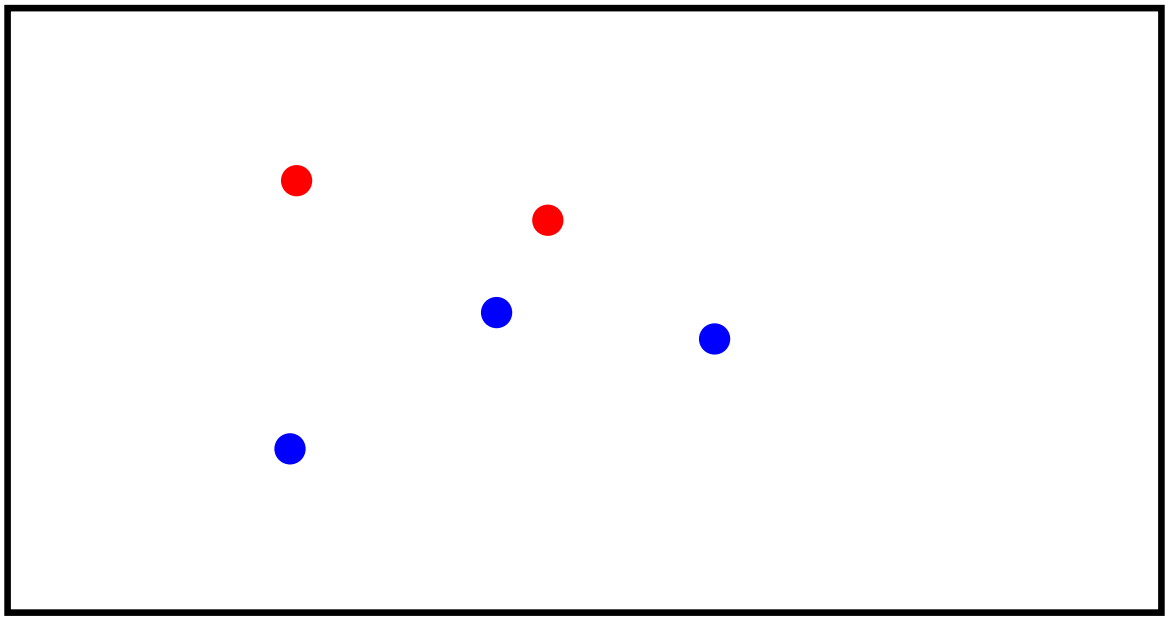

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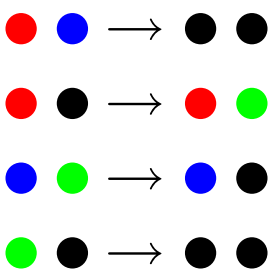

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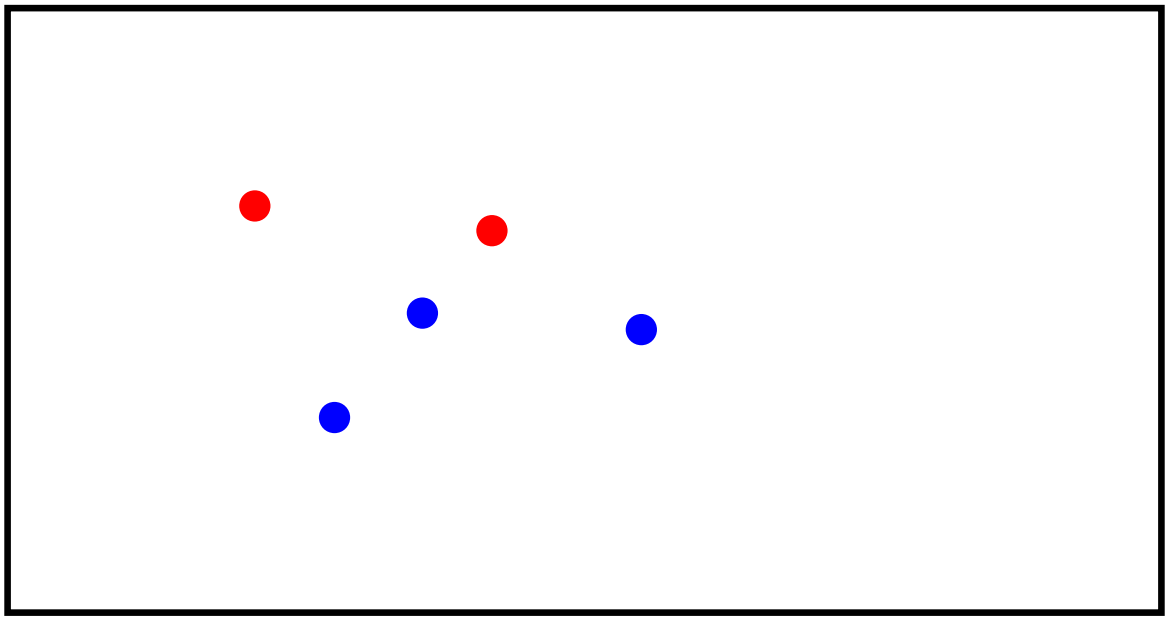

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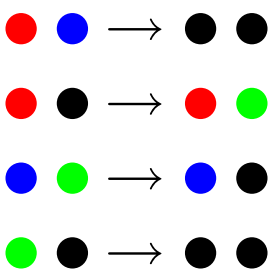

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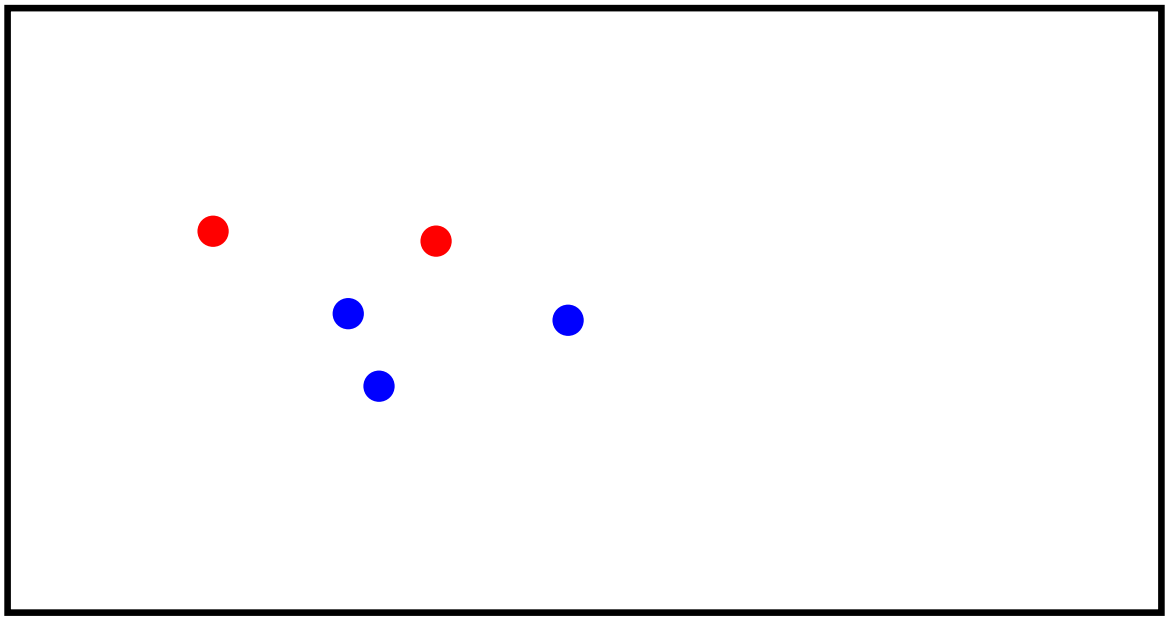

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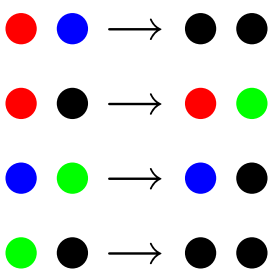

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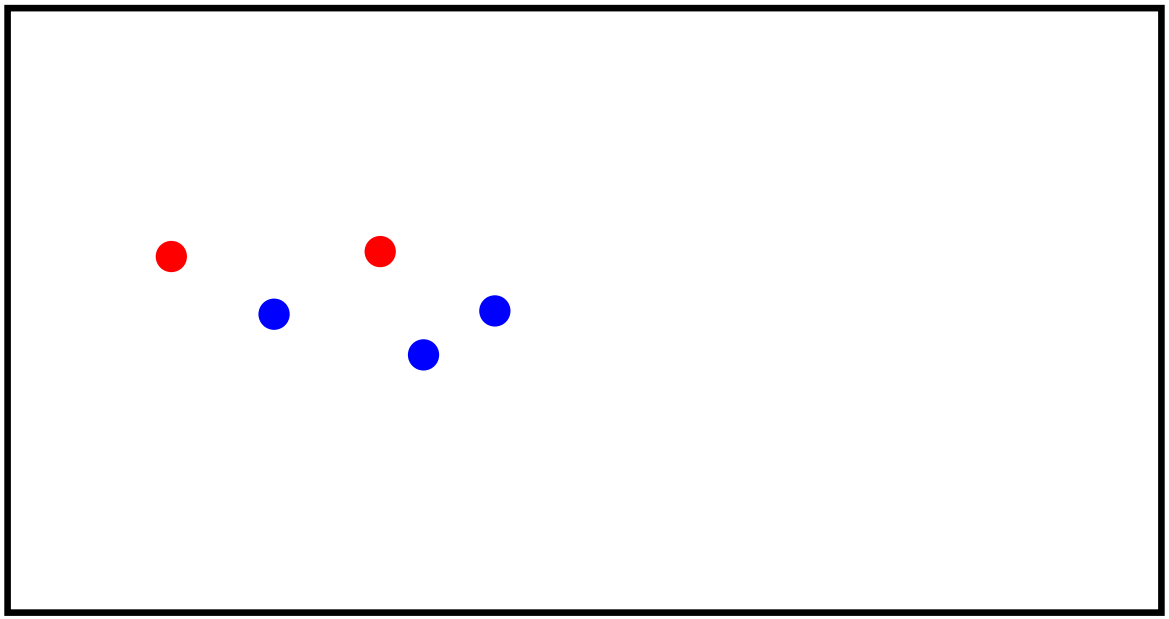

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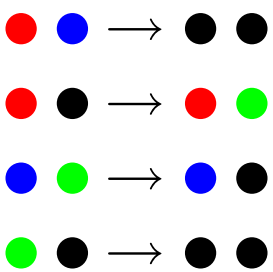

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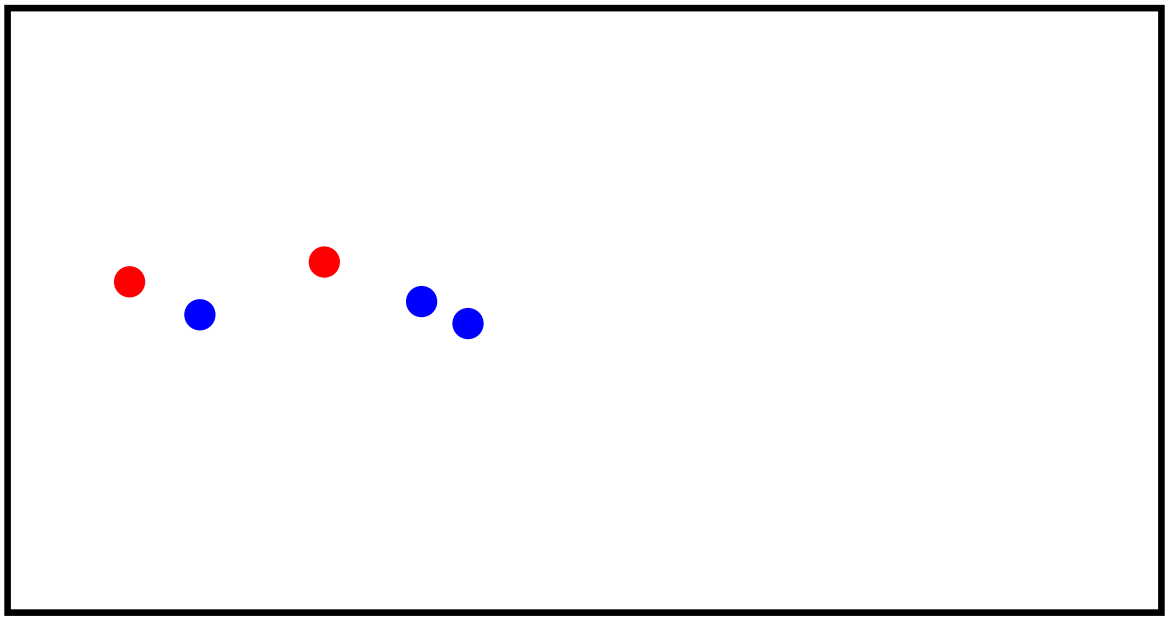

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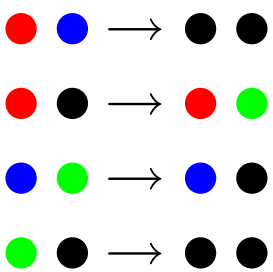

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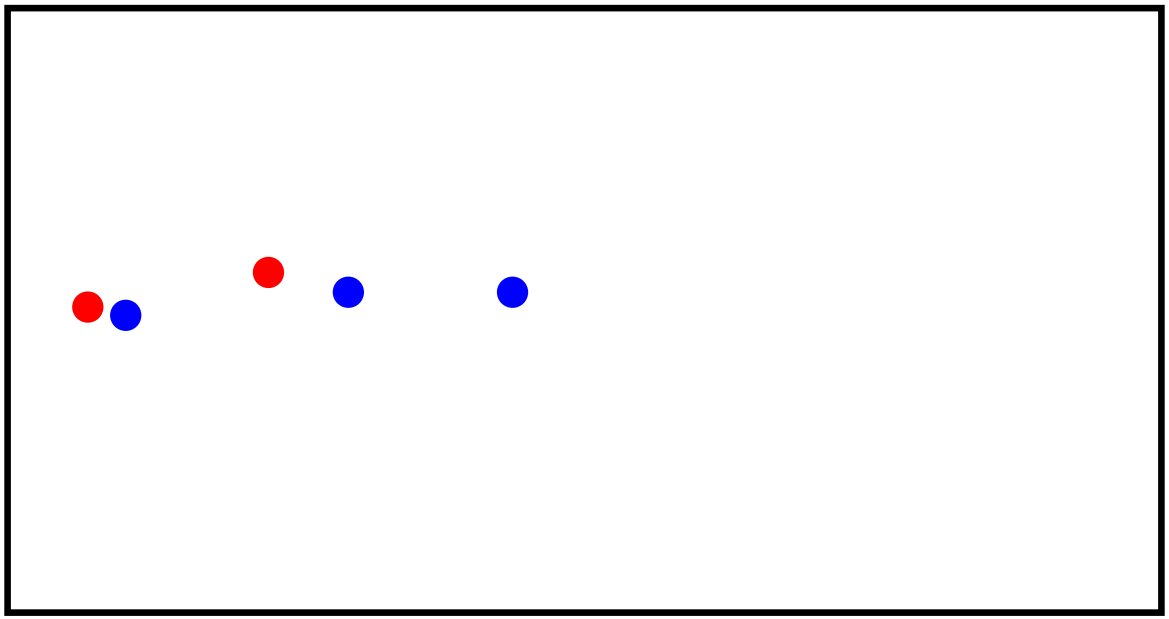

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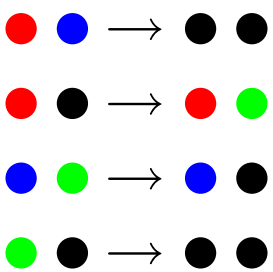

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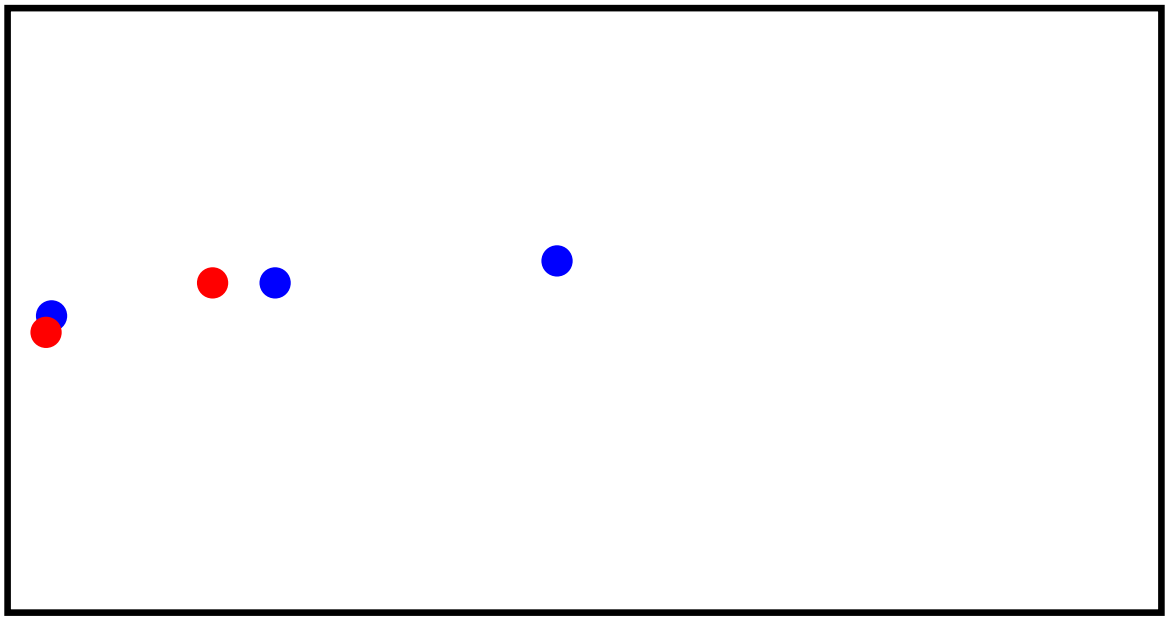

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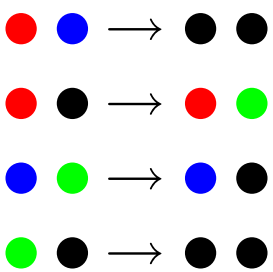

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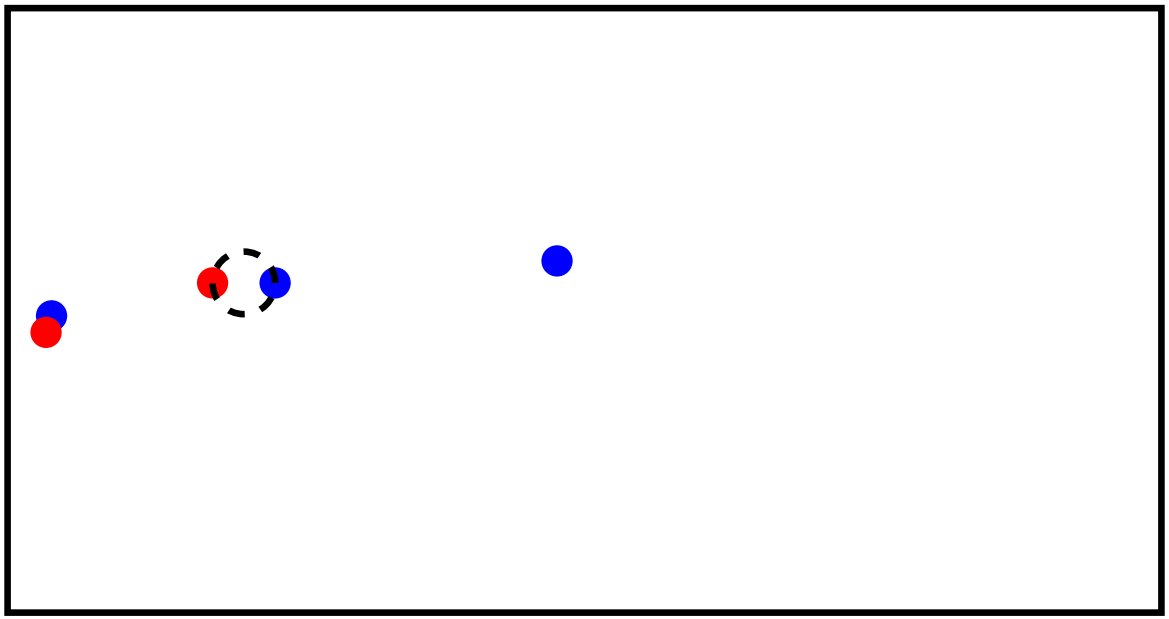

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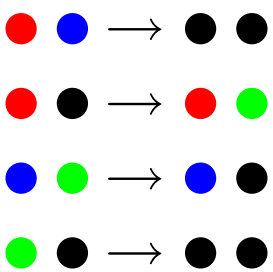

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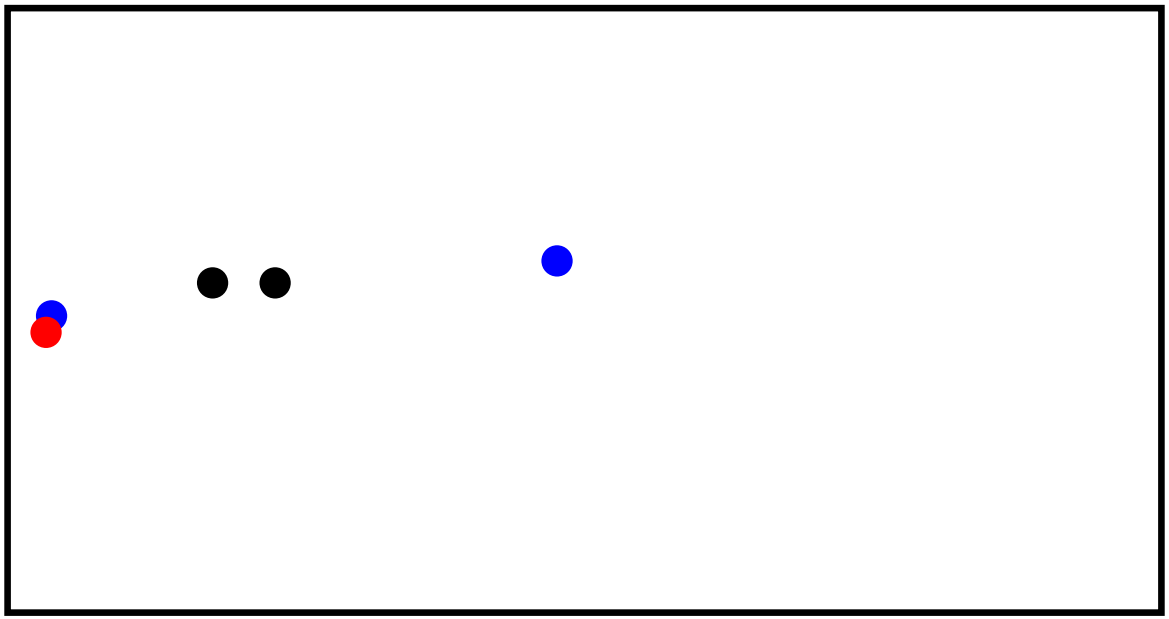

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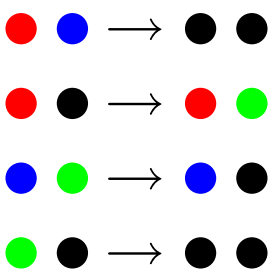

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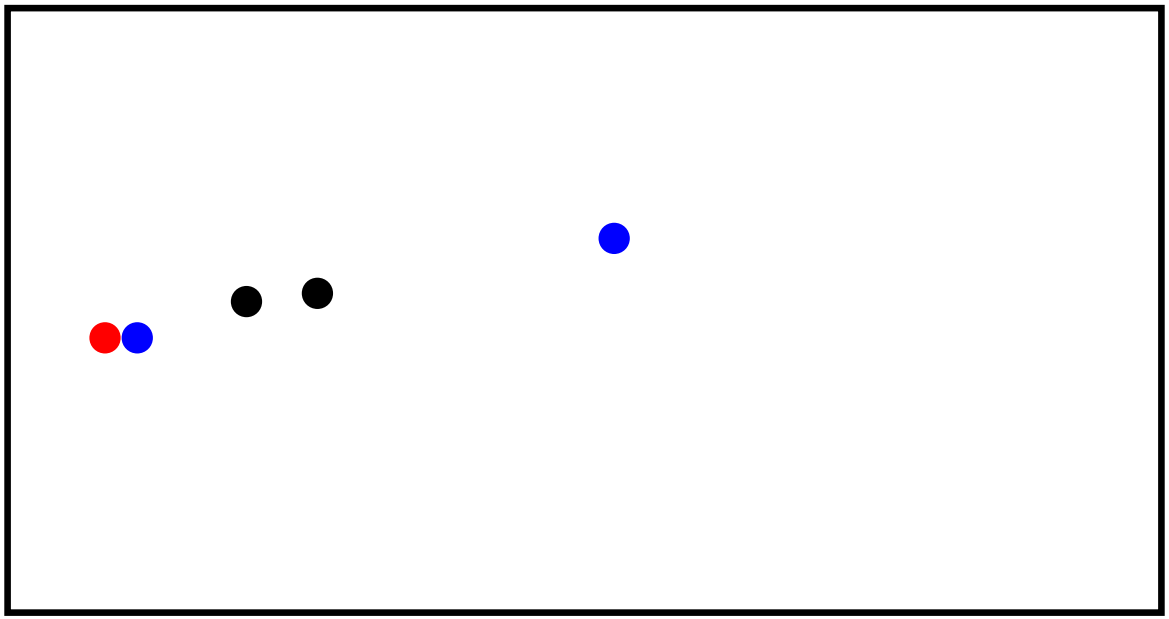

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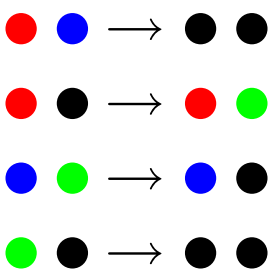

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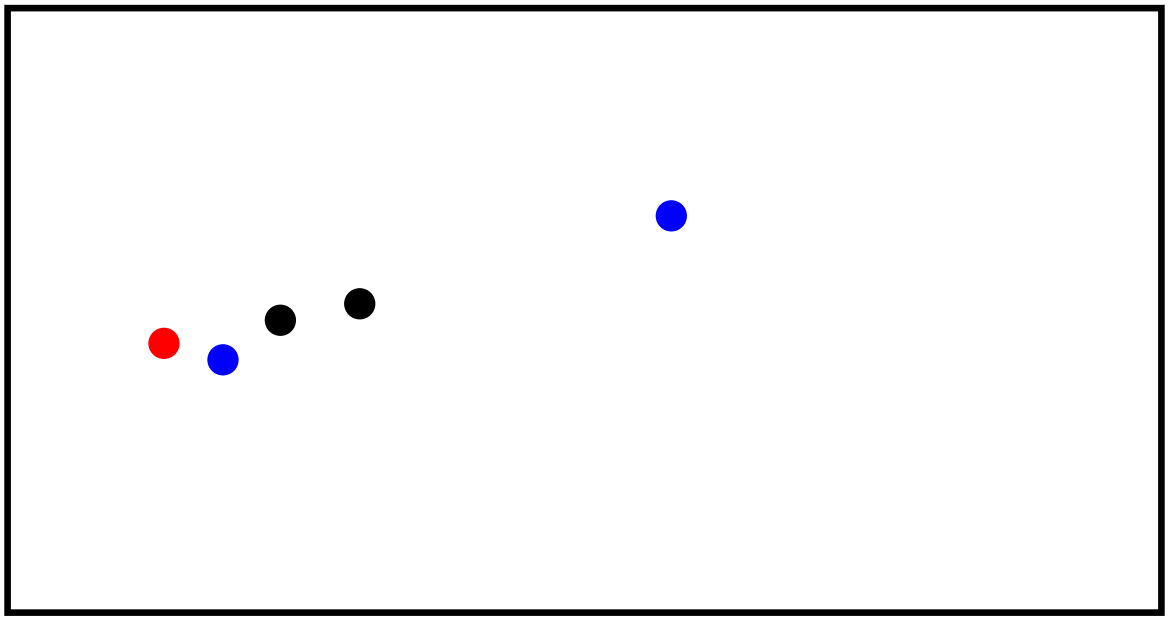

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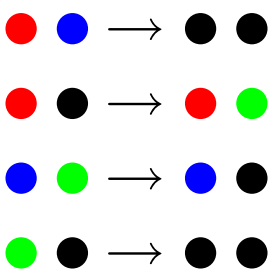

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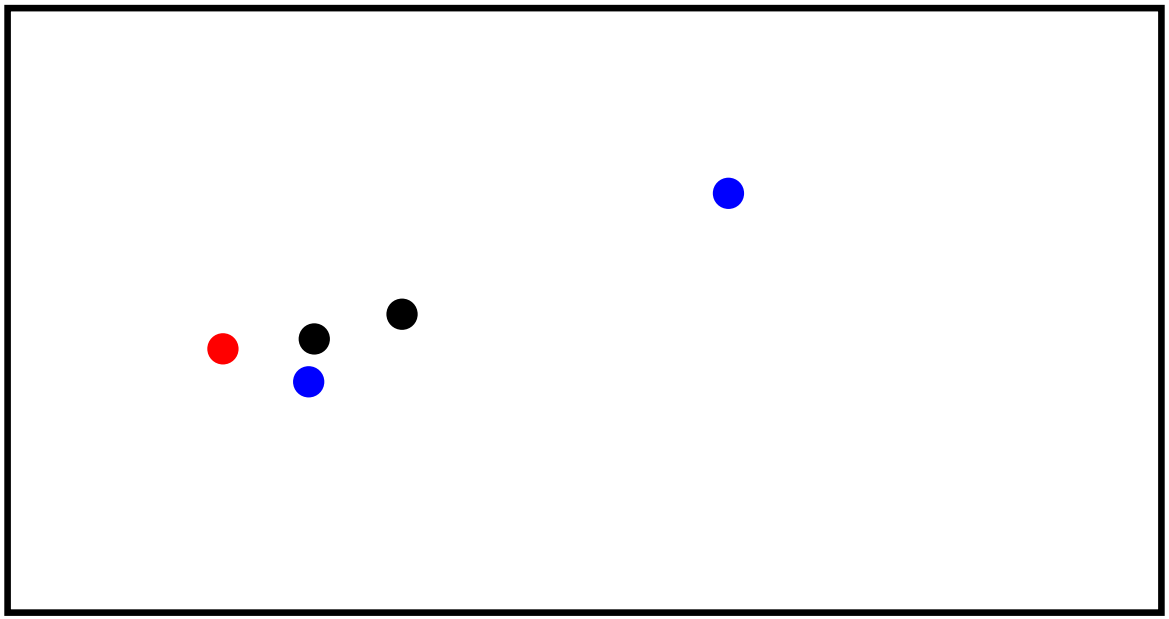

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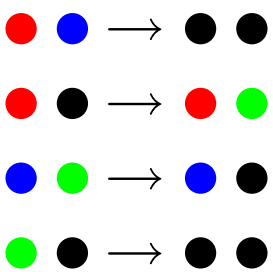

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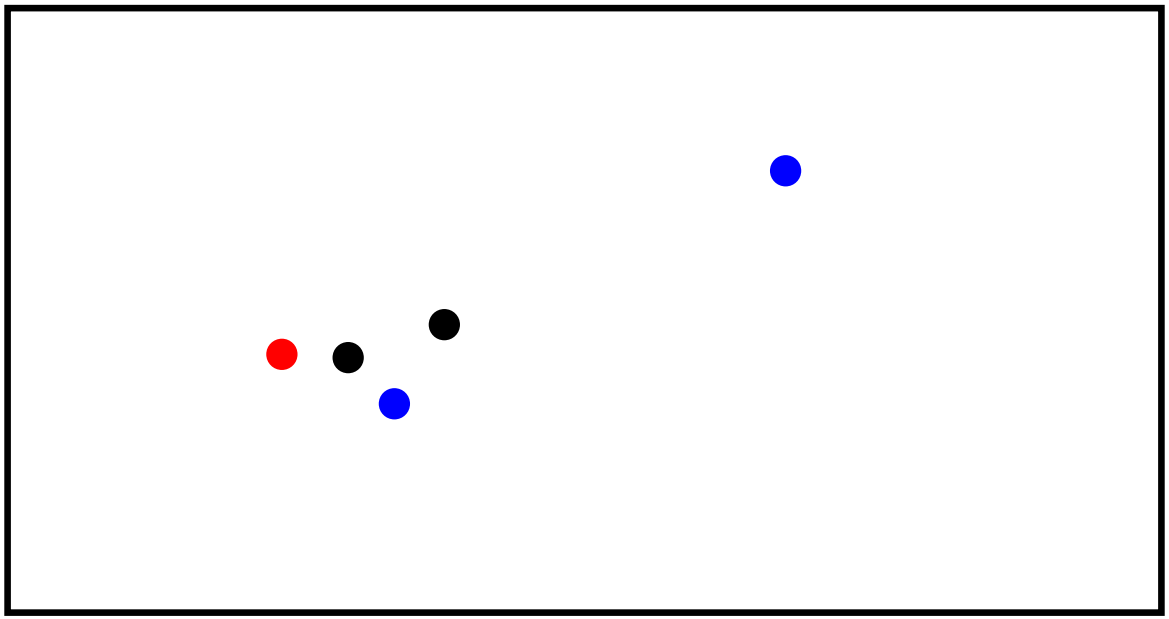

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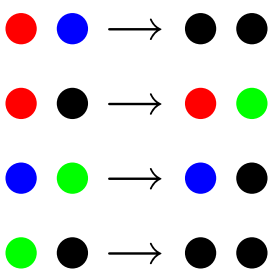

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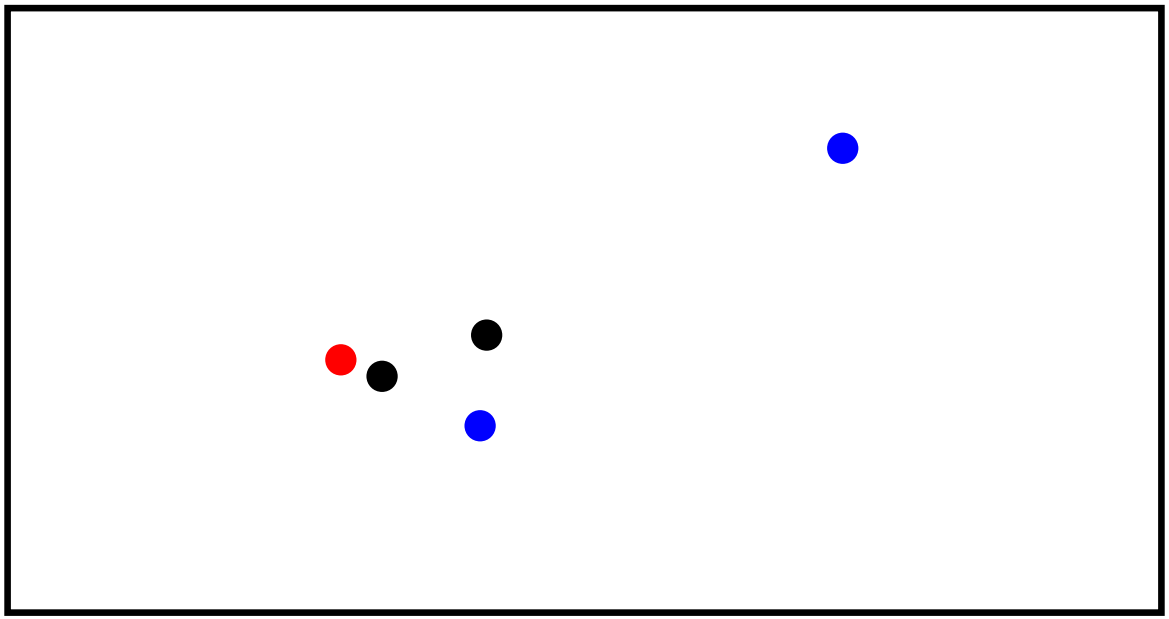

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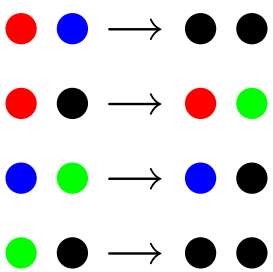

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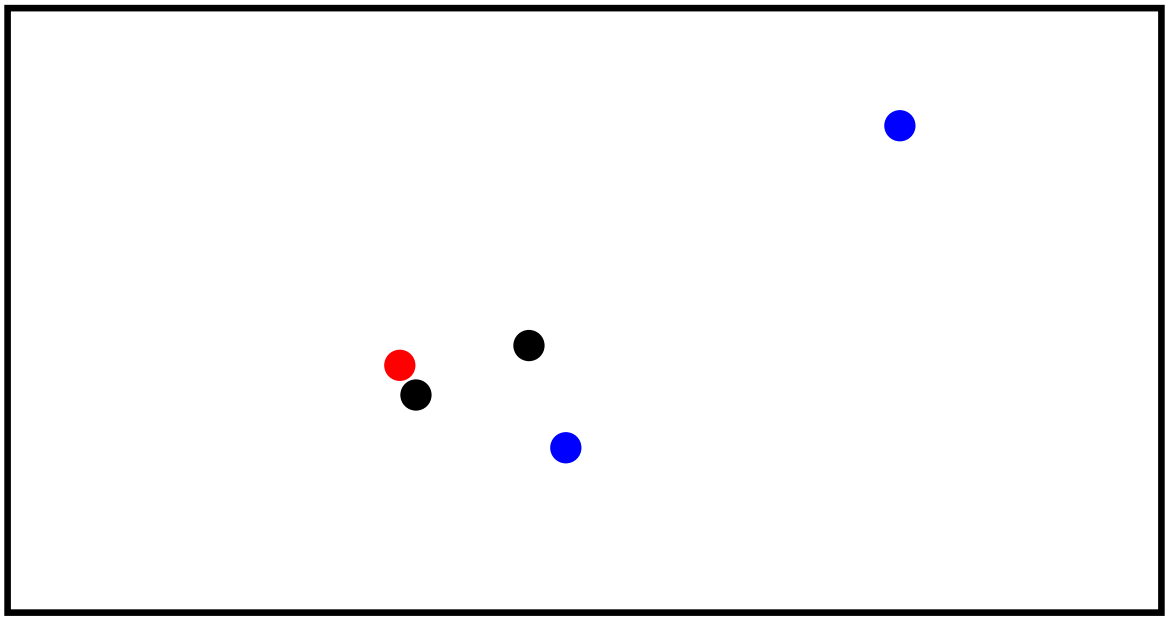

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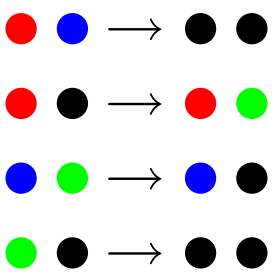

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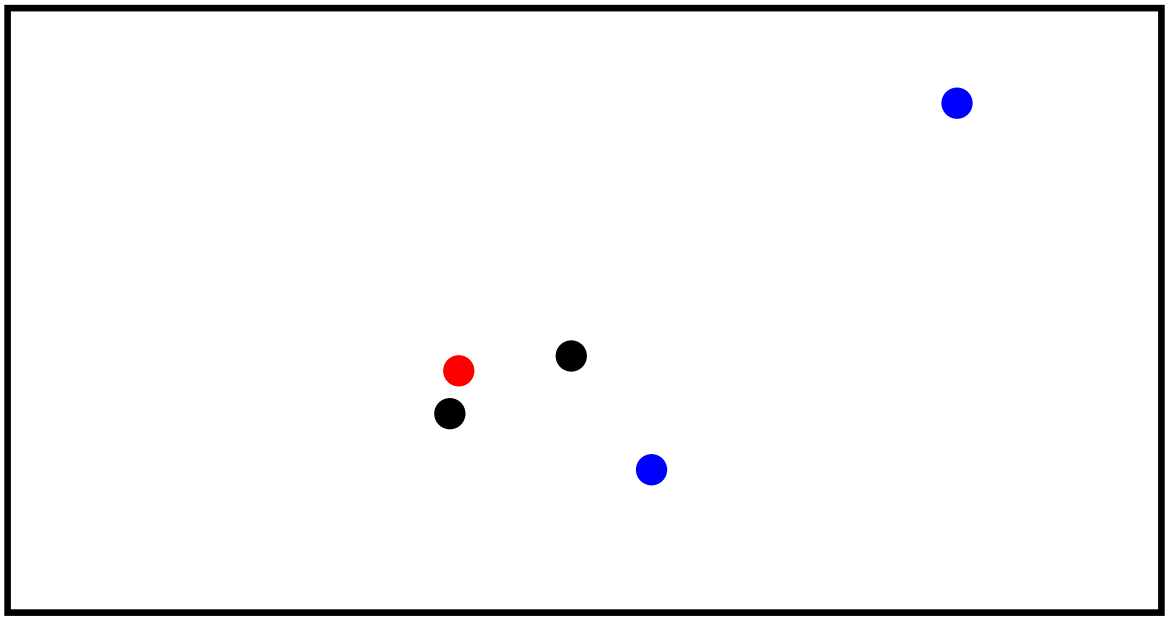

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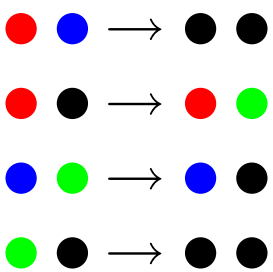

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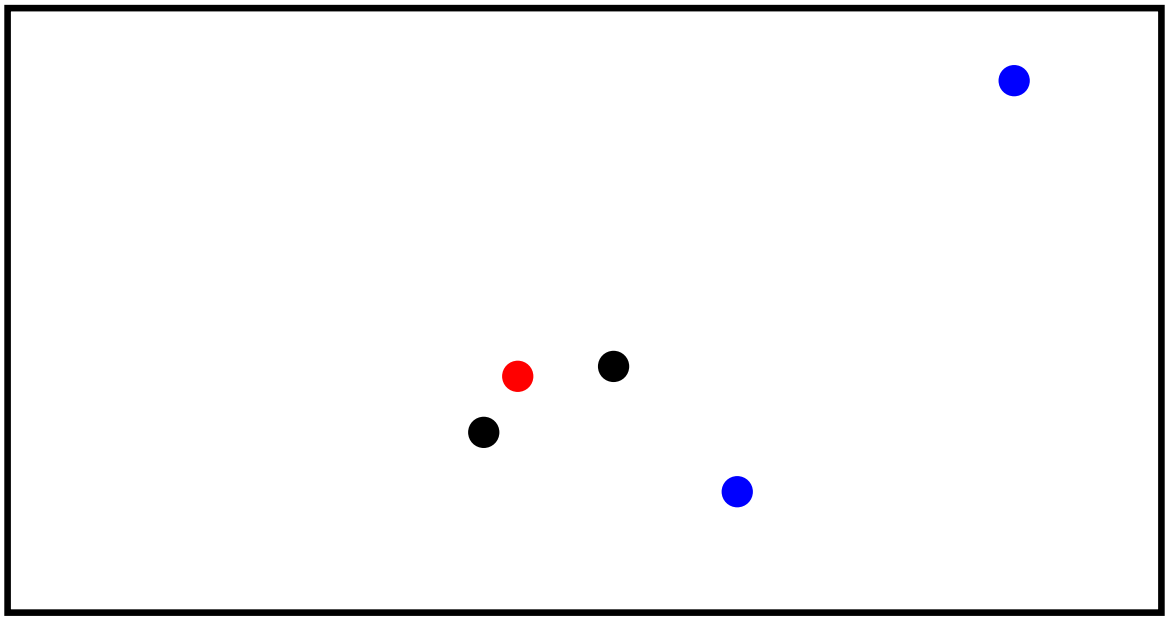

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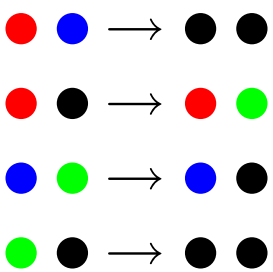

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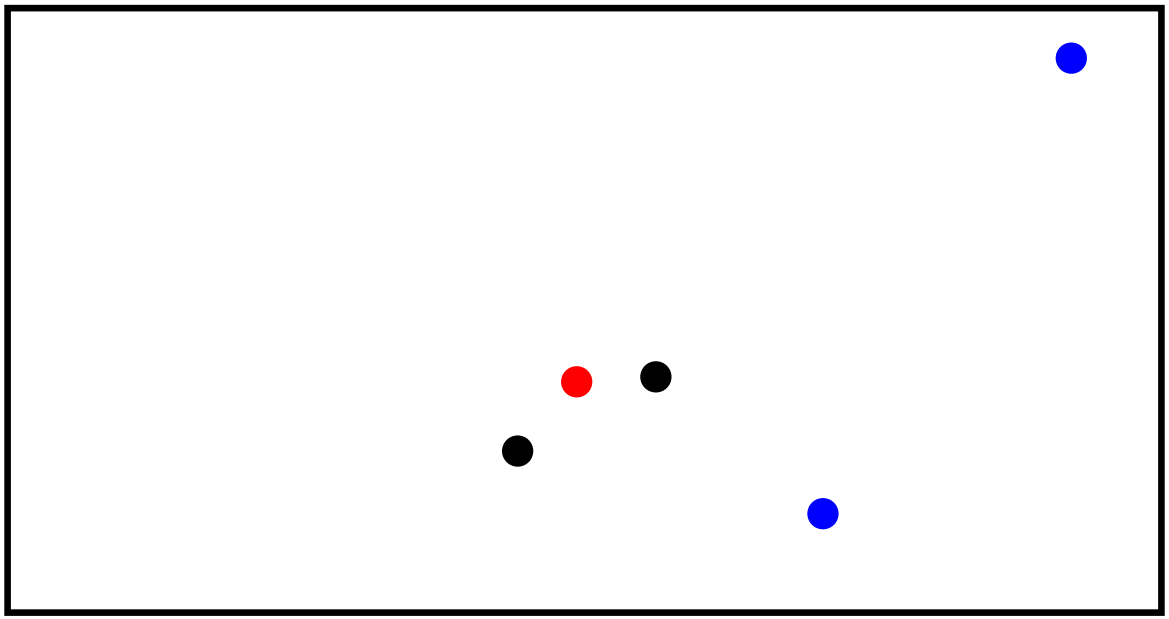

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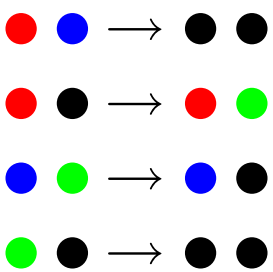

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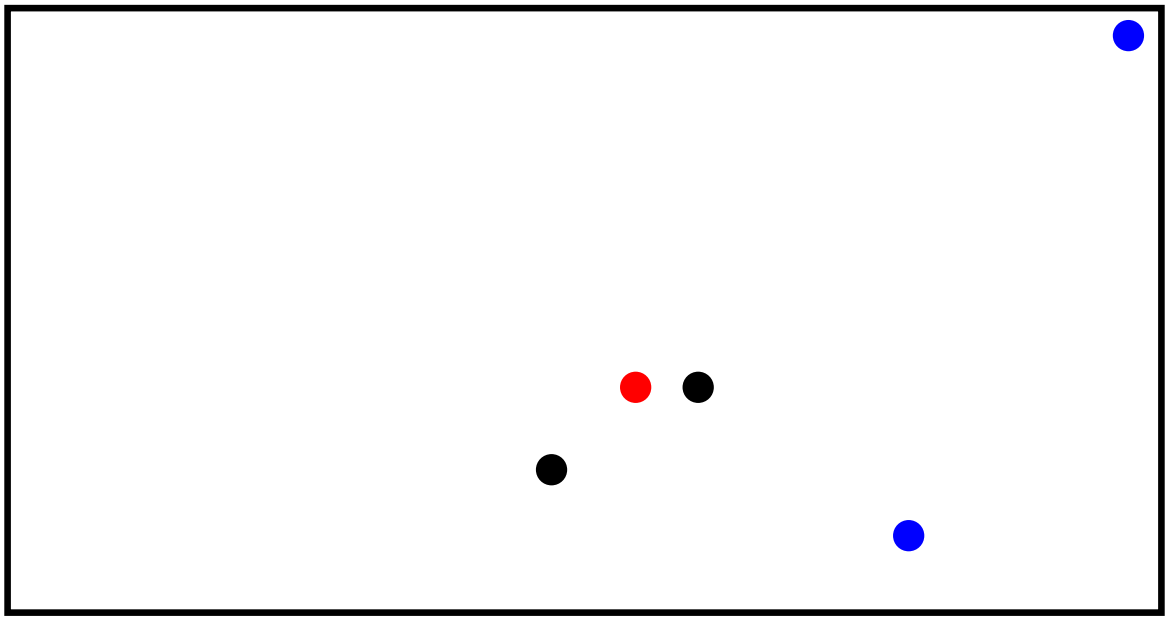

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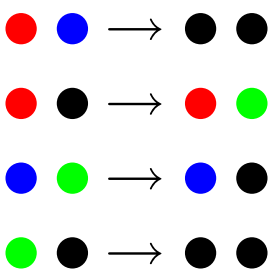

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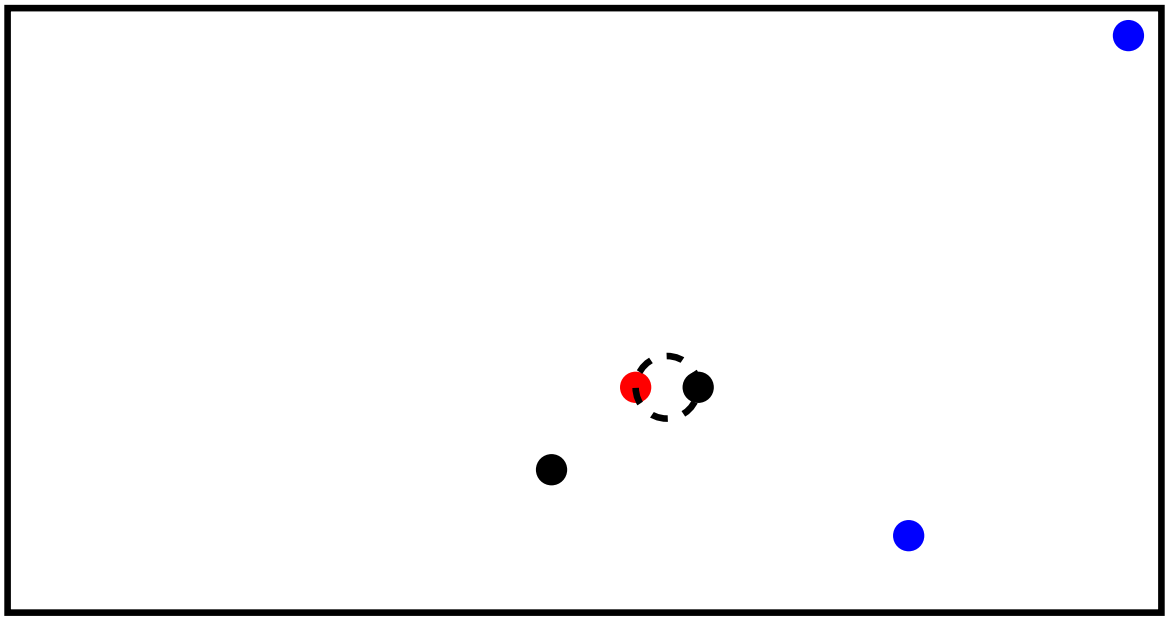

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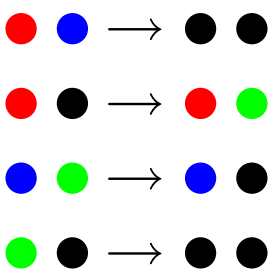

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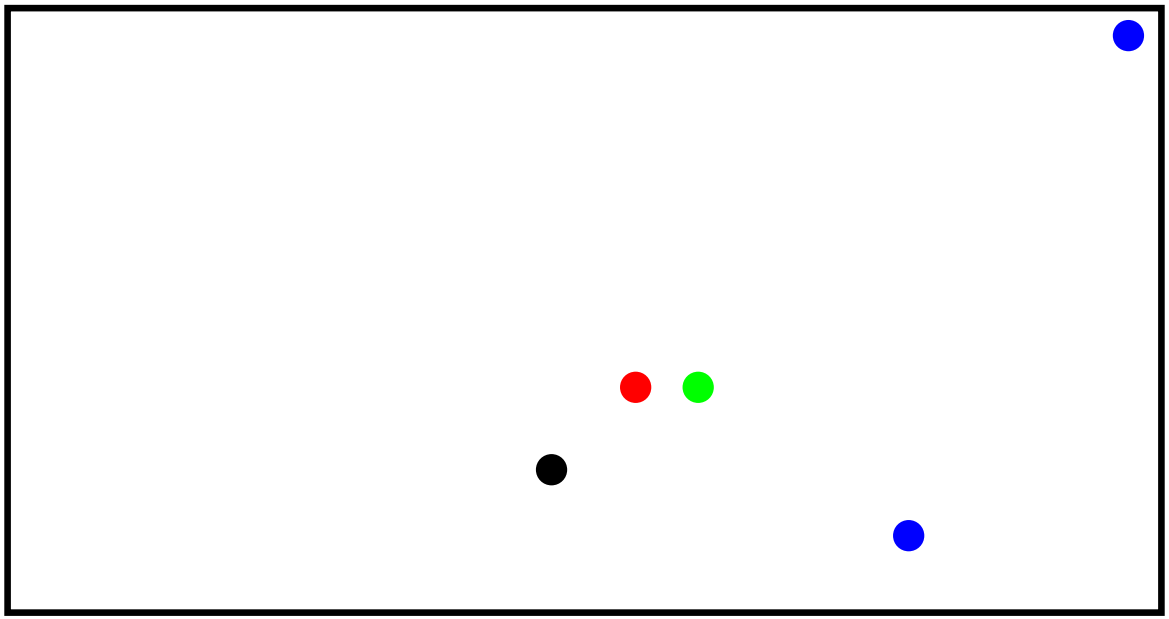

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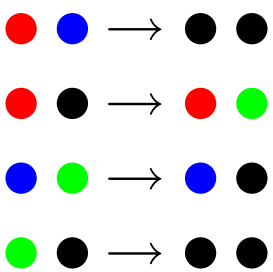

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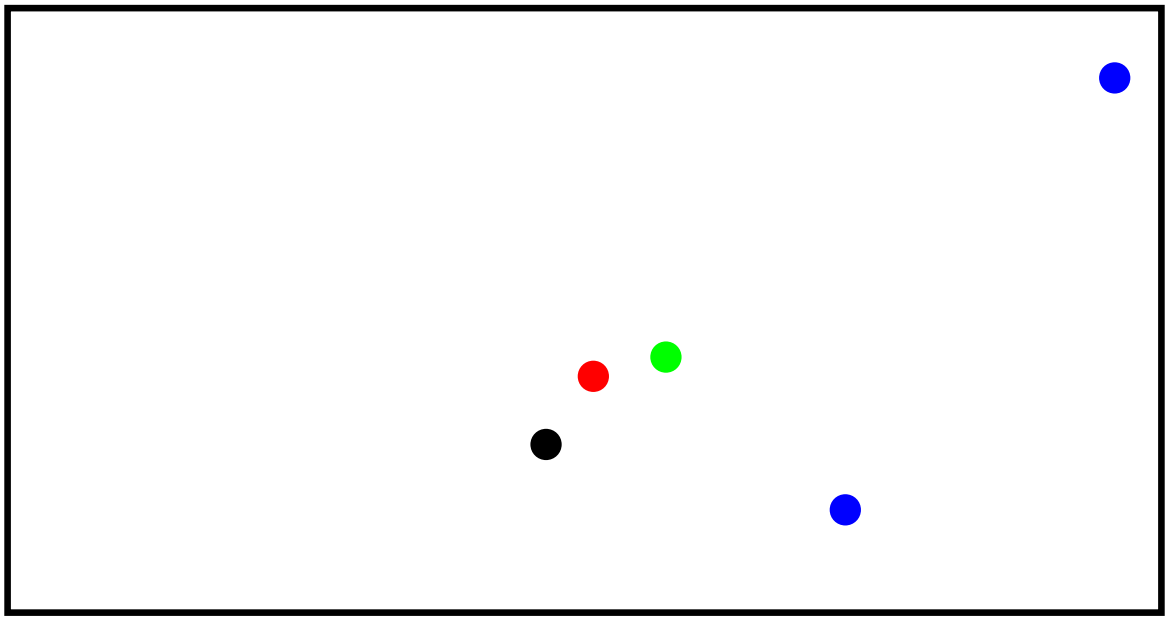

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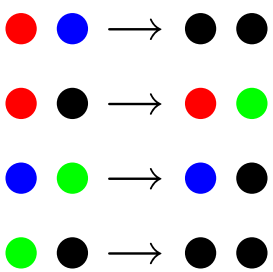

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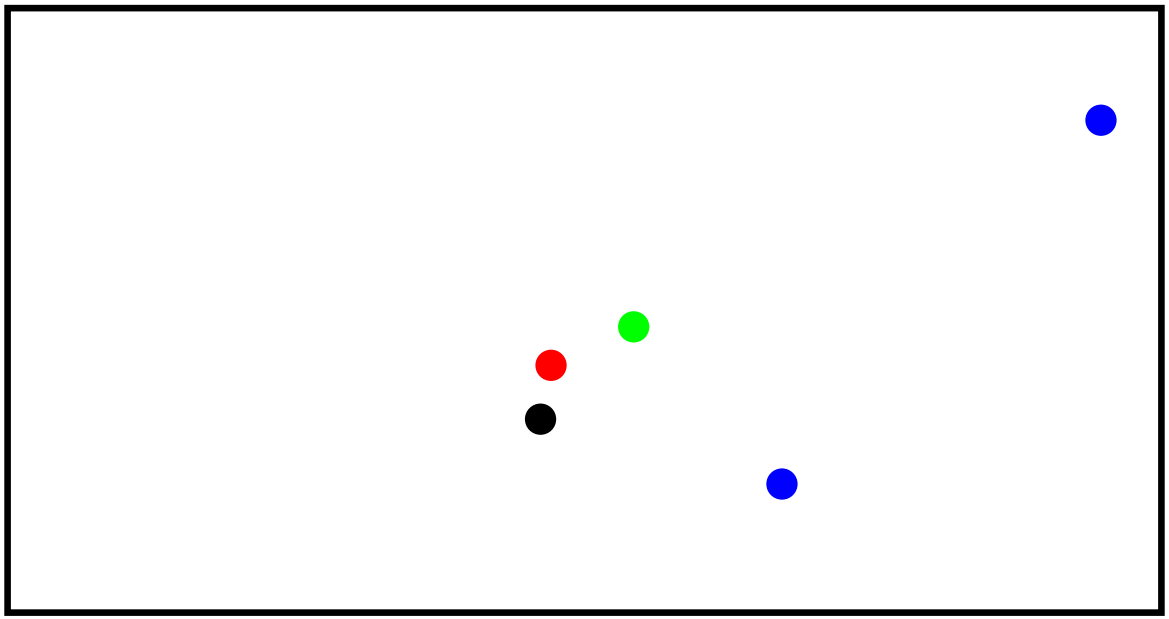

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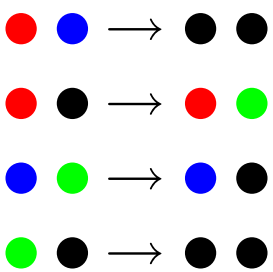

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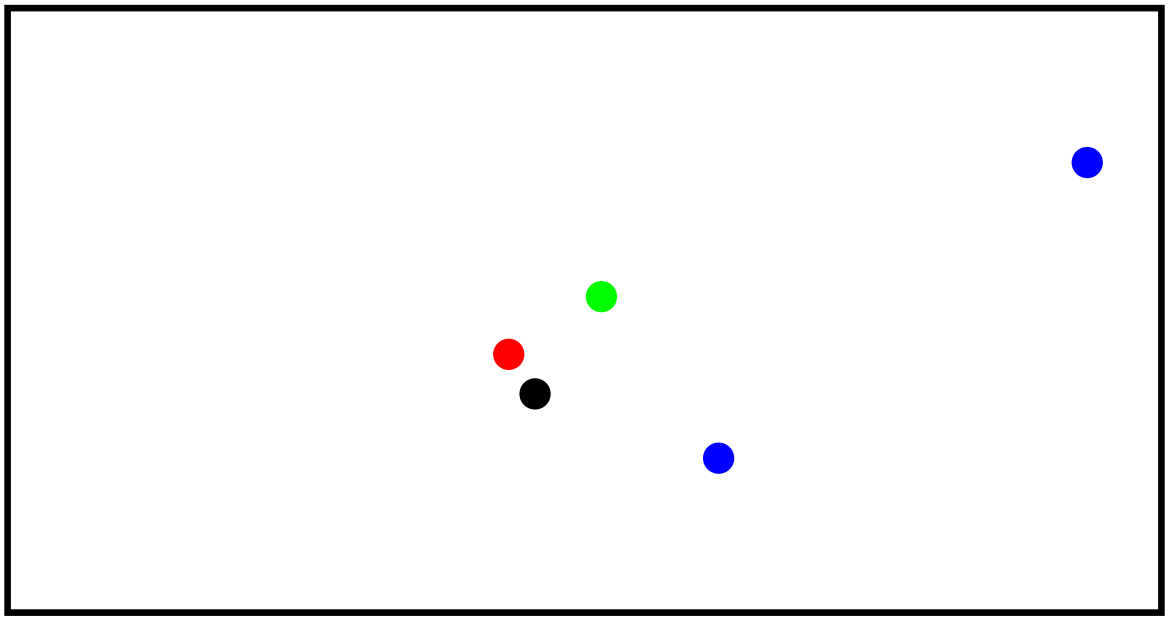

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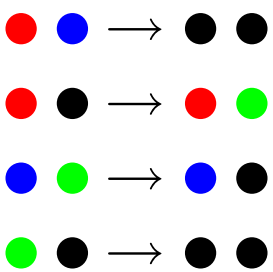

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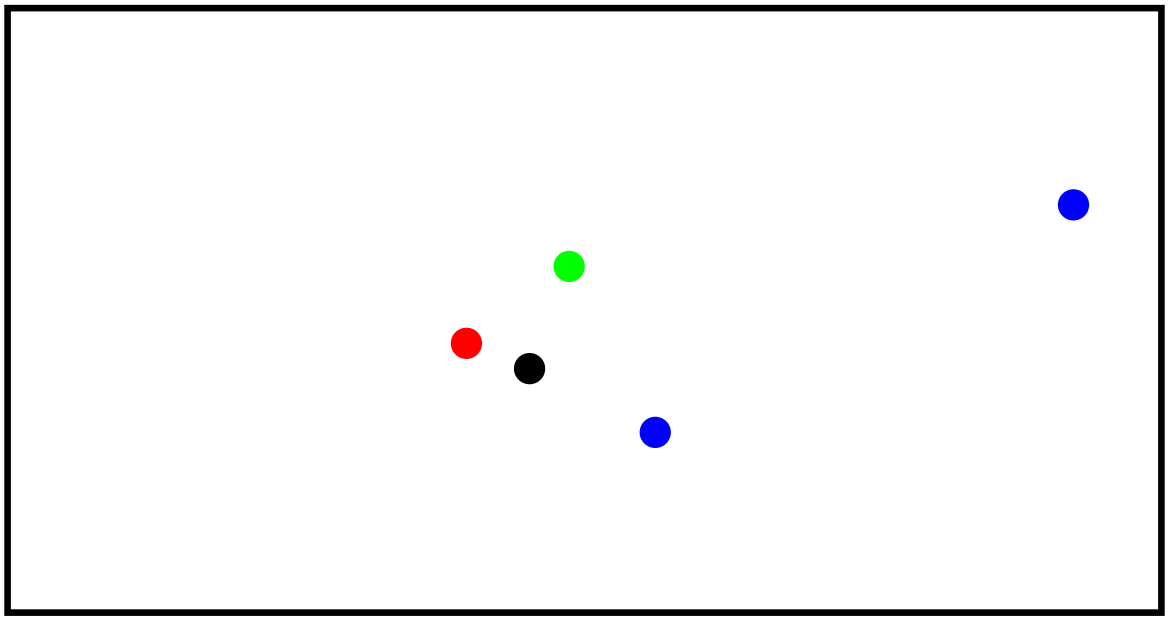

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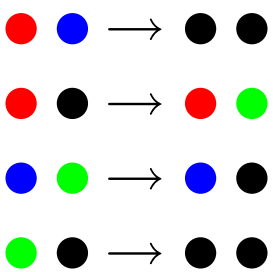

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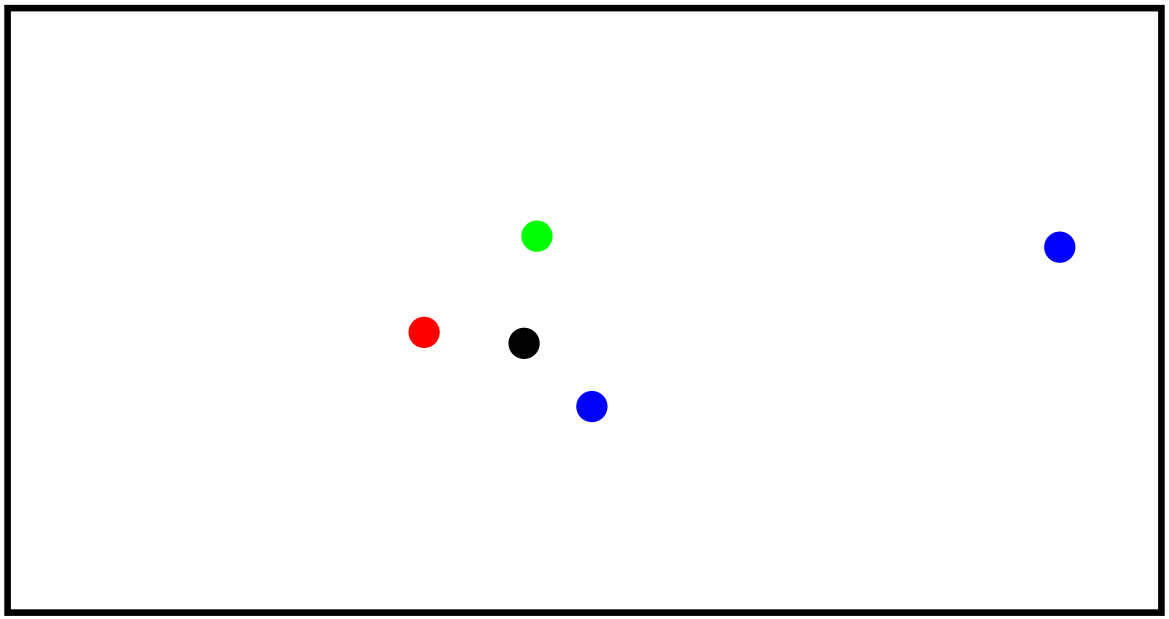

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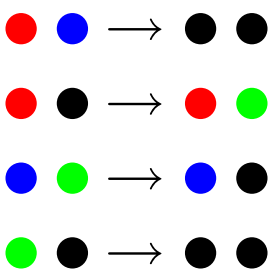
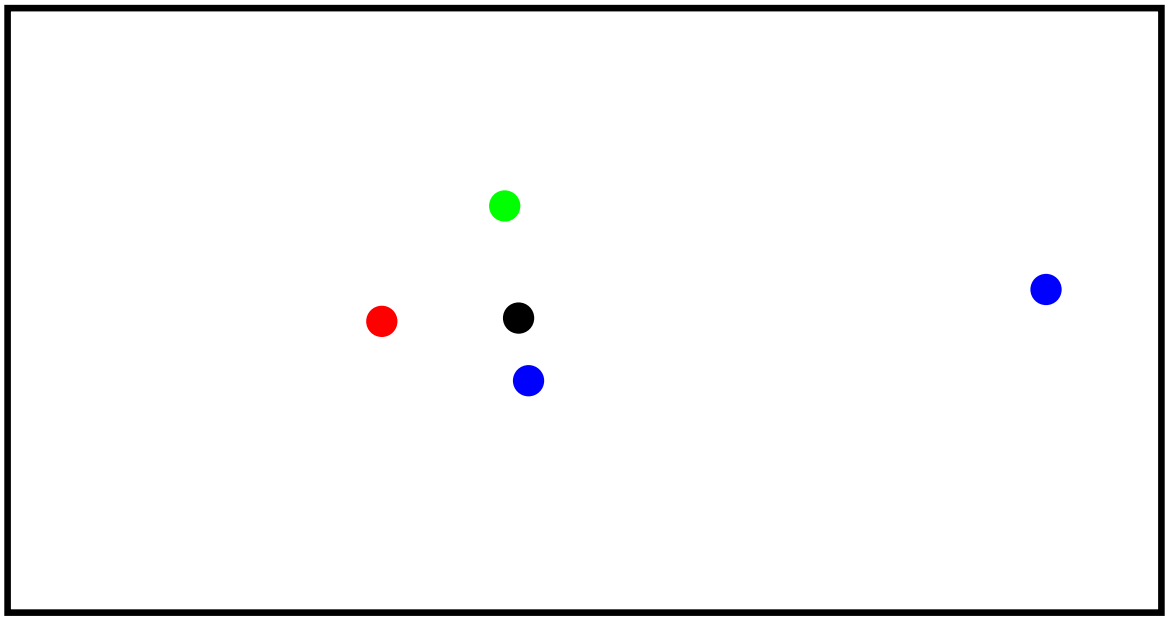


Example 1: starting with 3 blues, 2red



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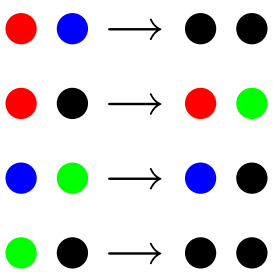
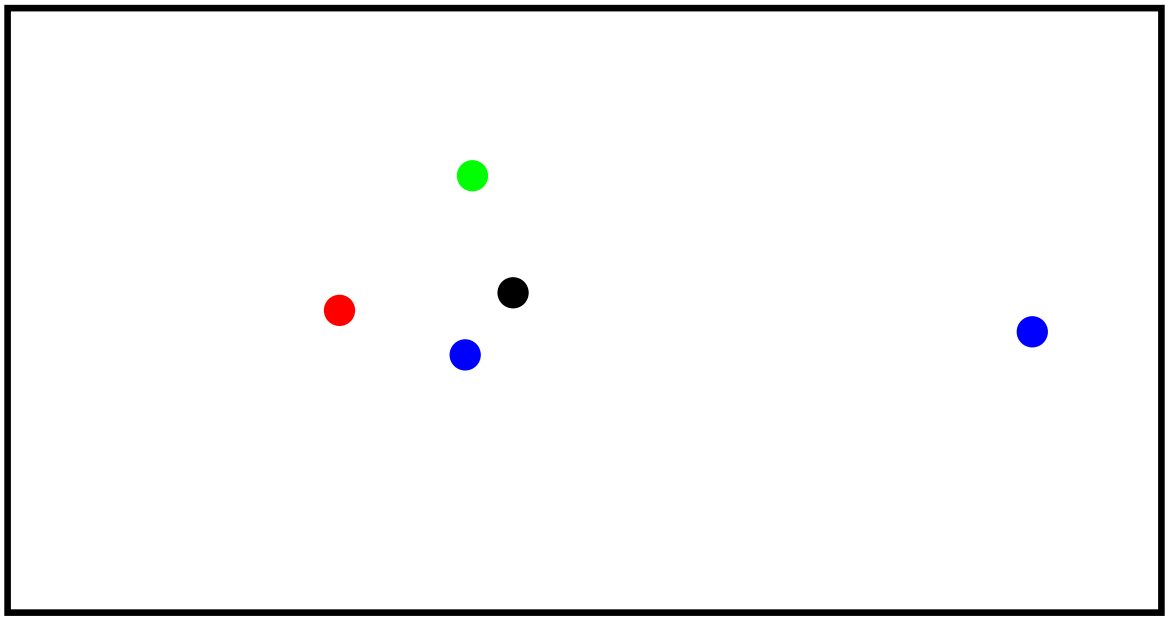


Example 1: starting with 3 blues, 2red



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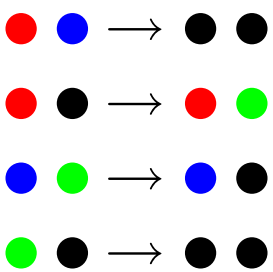
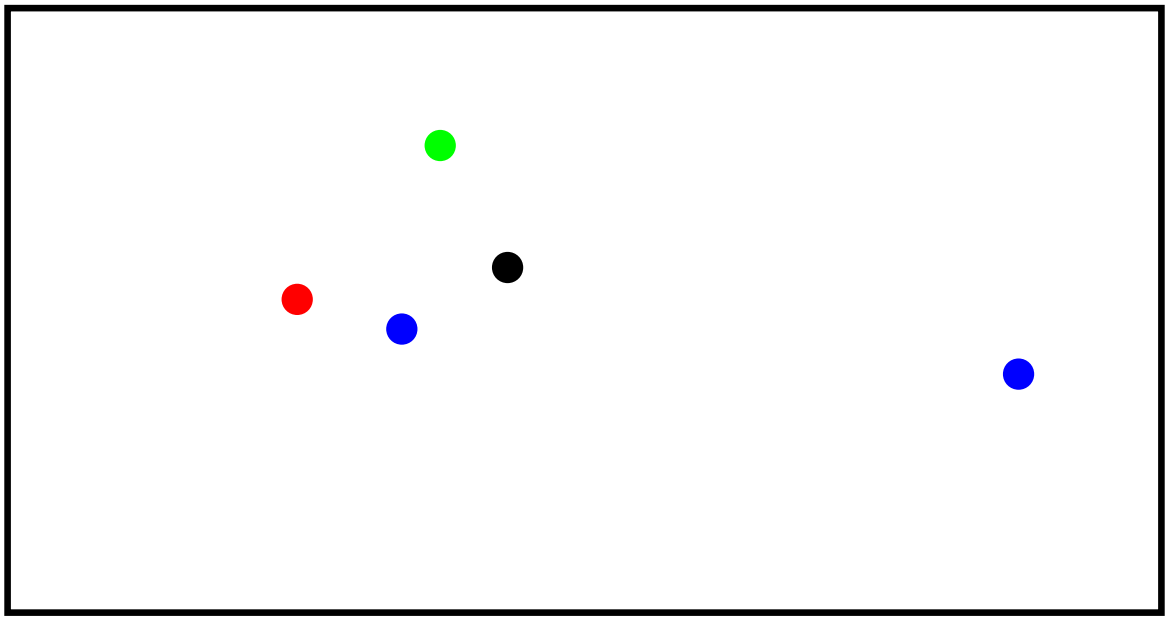


Example 1: starting with 3 blues, 2red



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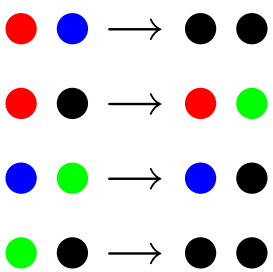
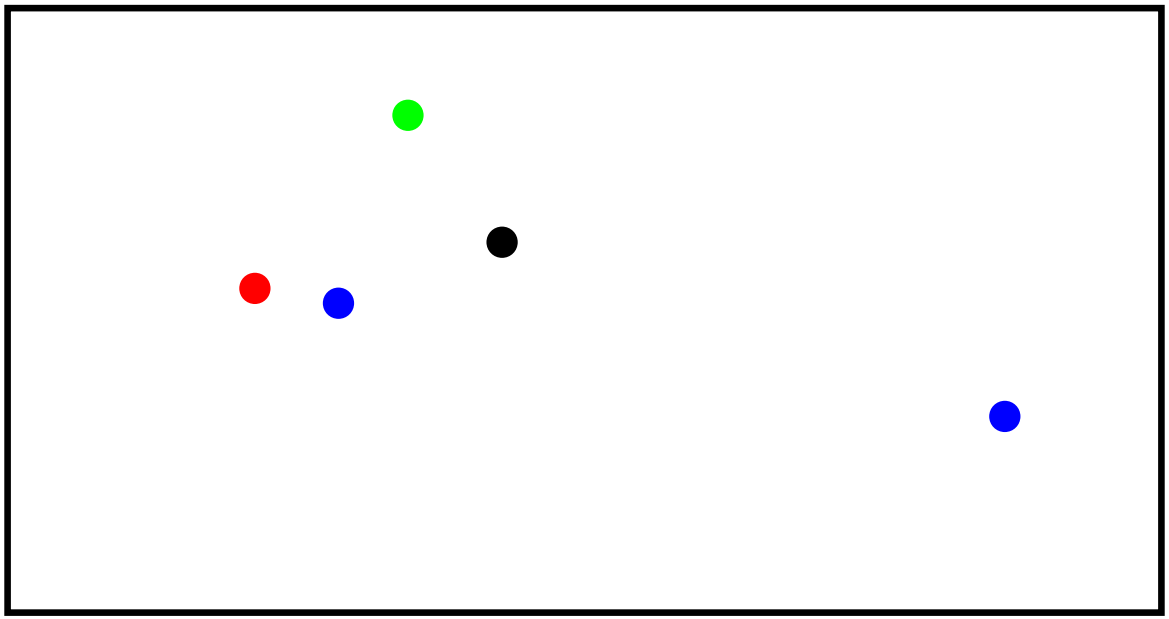


Example 1: starting with 3 blues, 2red



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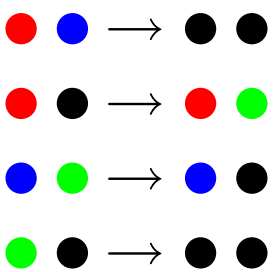

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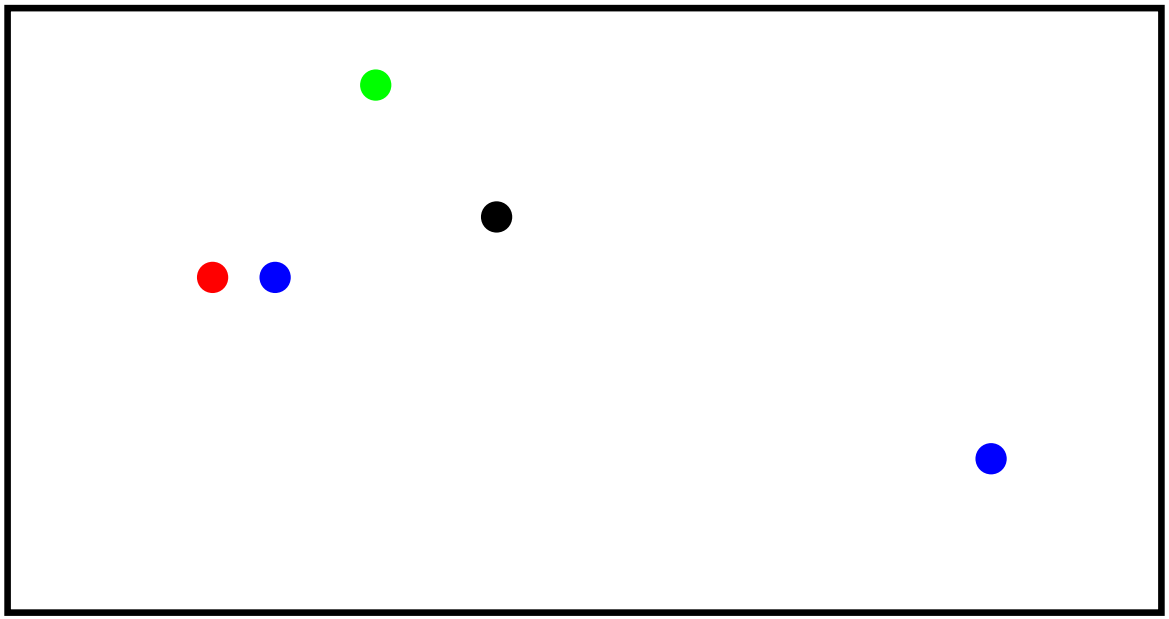

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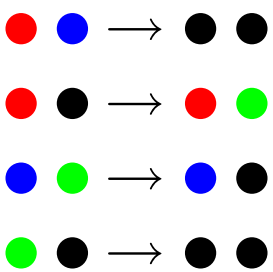

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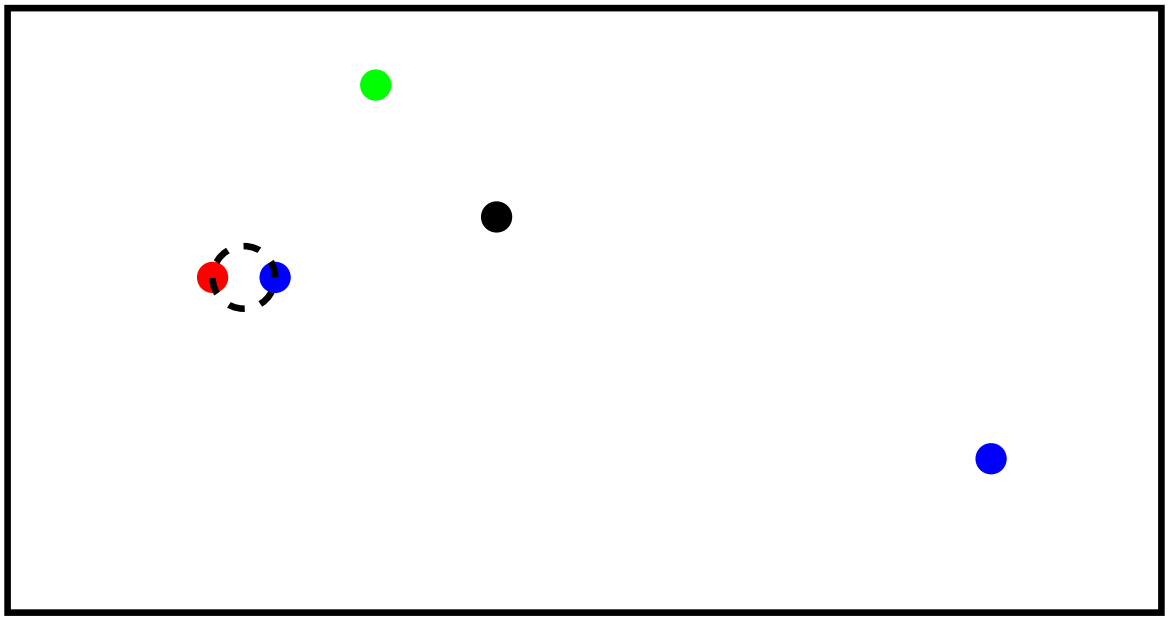

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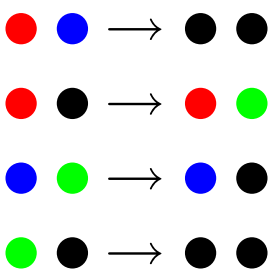

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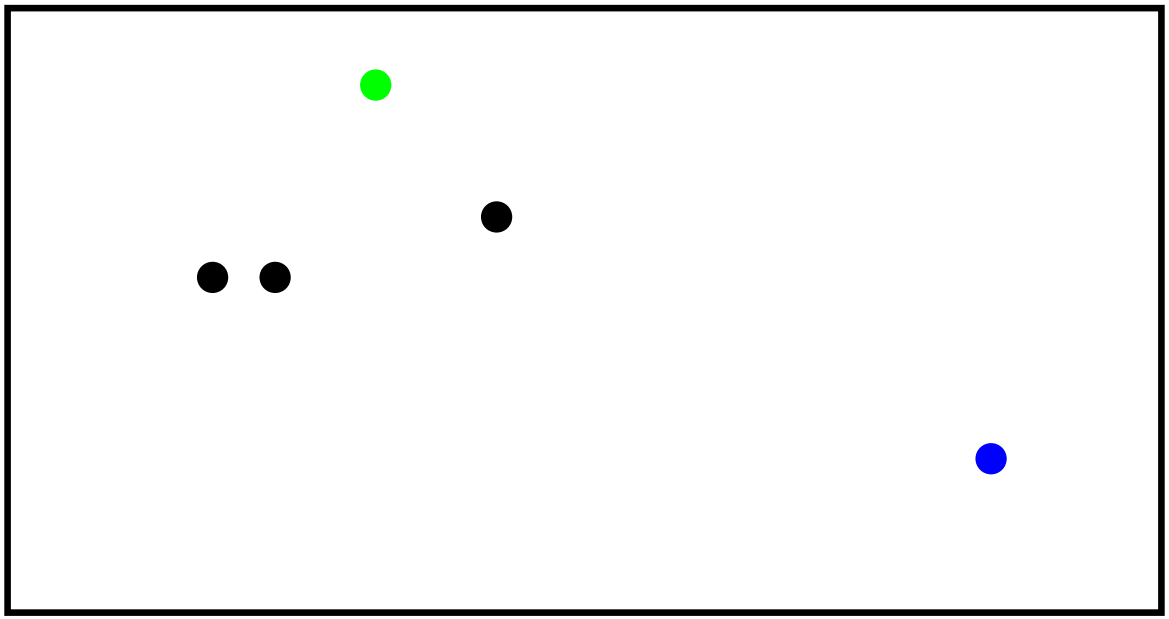

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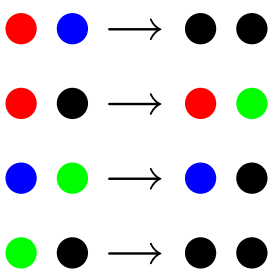

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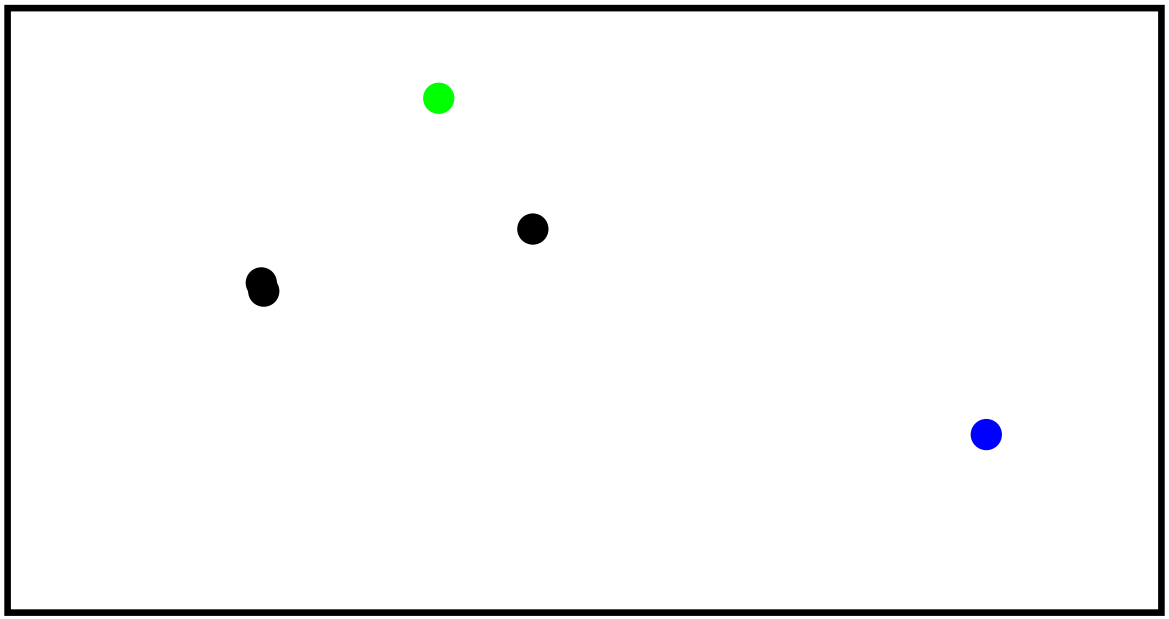

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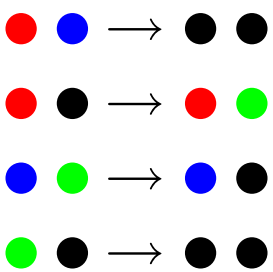

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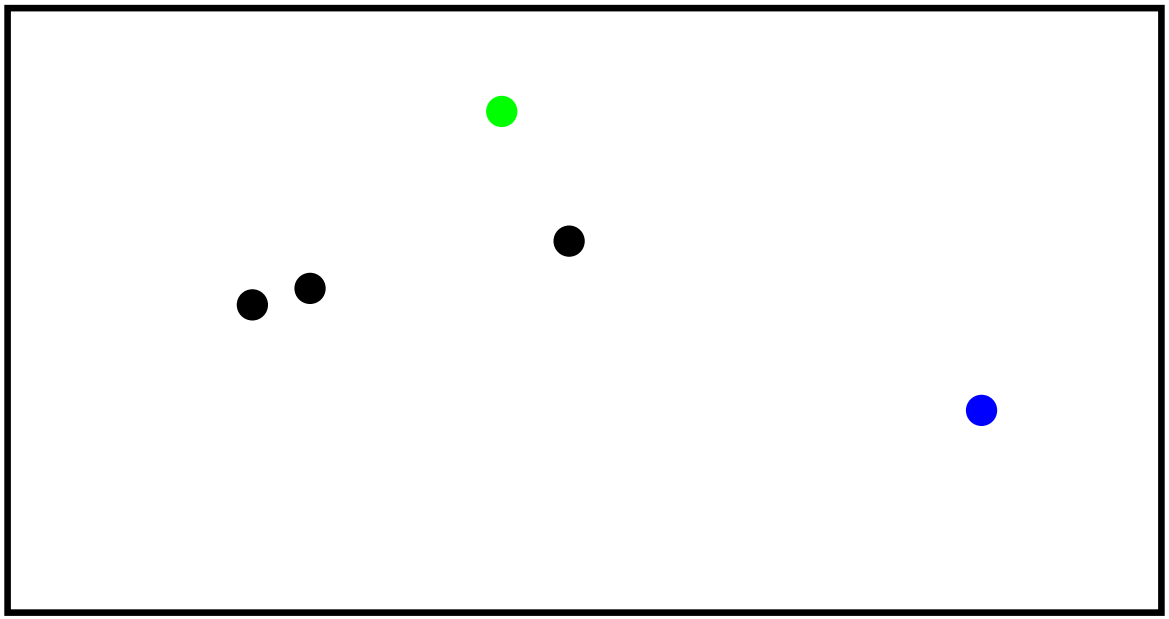

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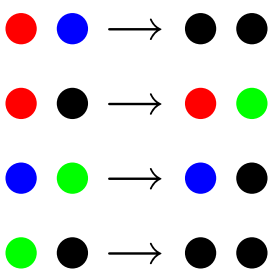

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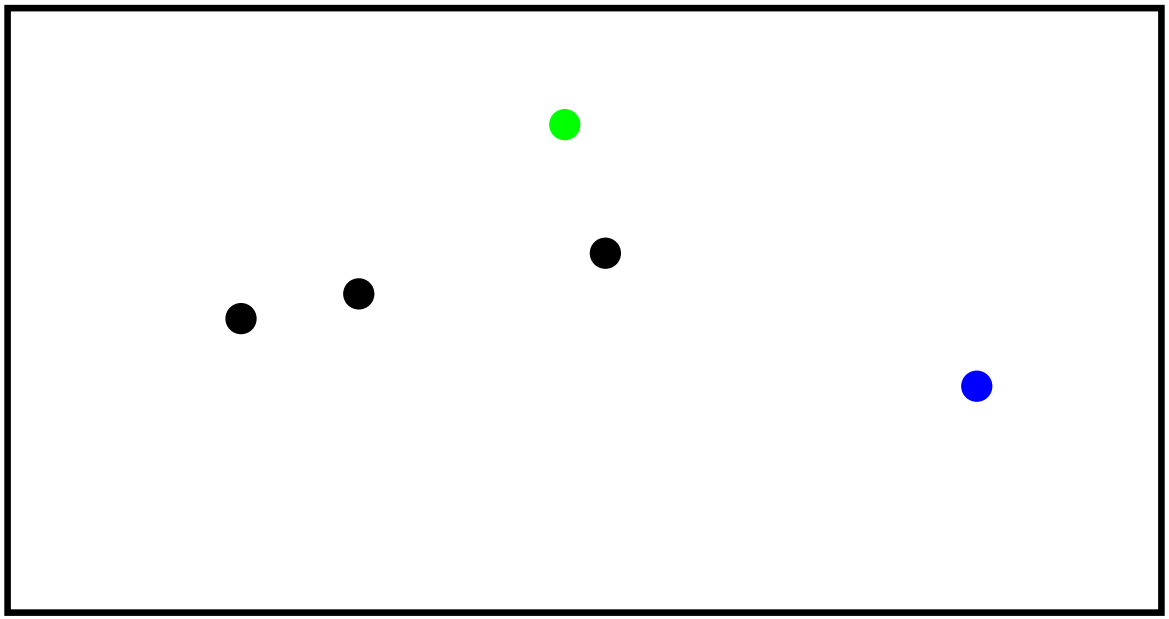

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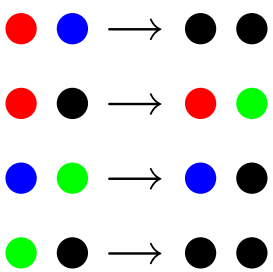

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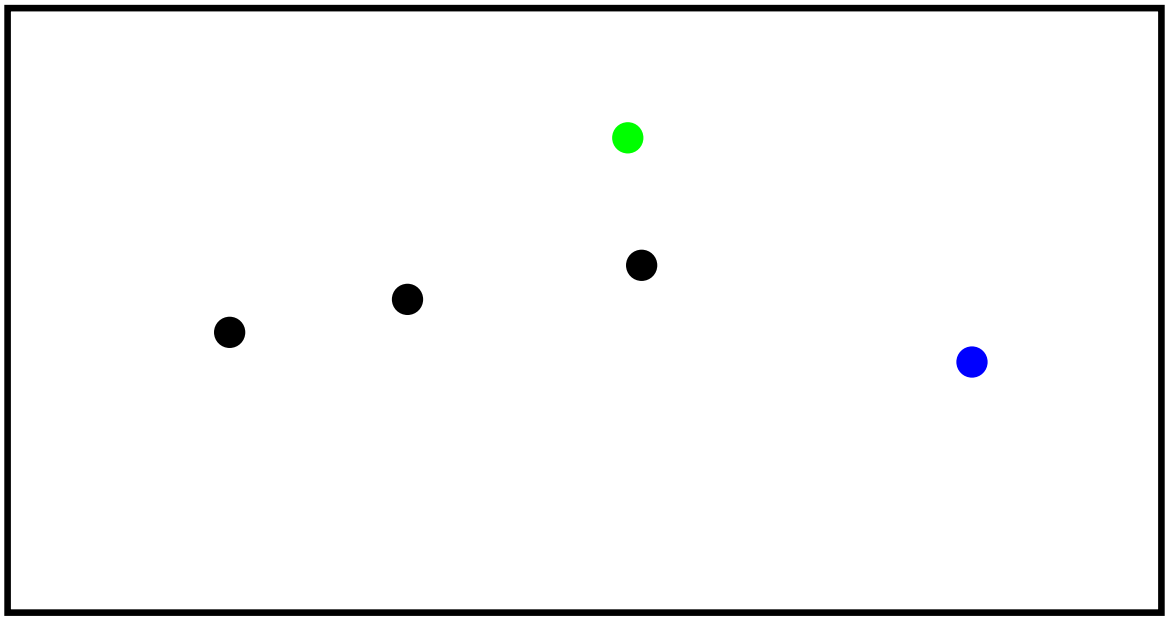

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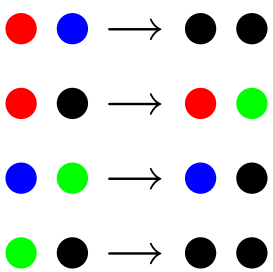

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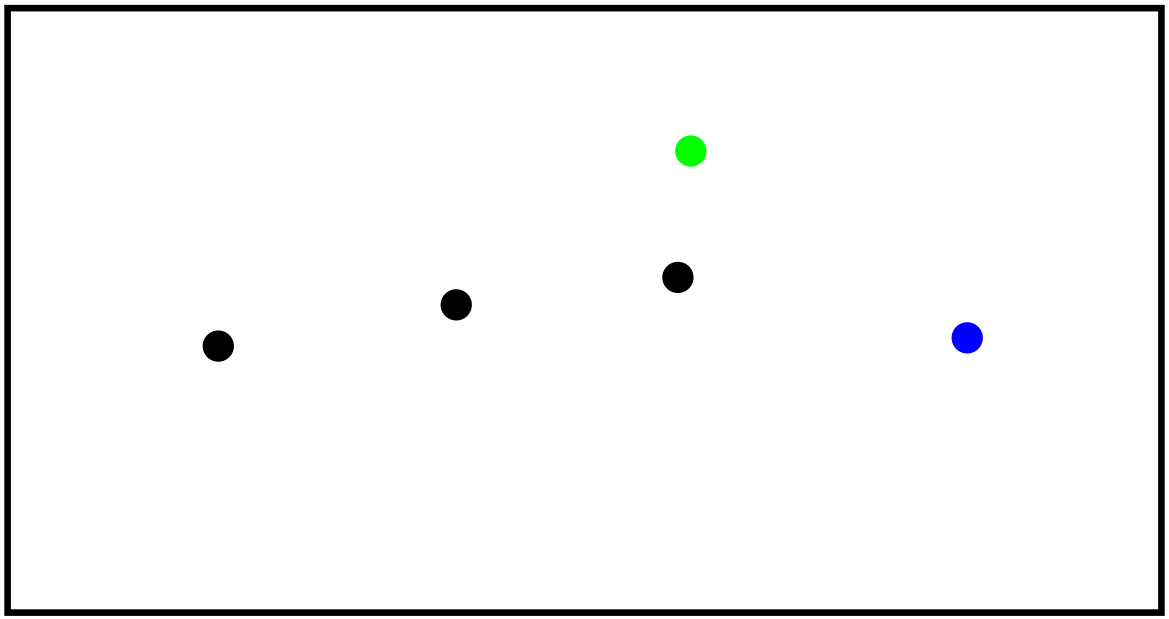

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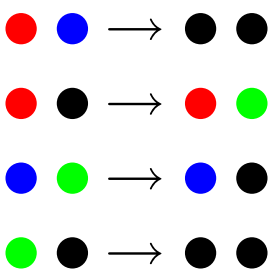
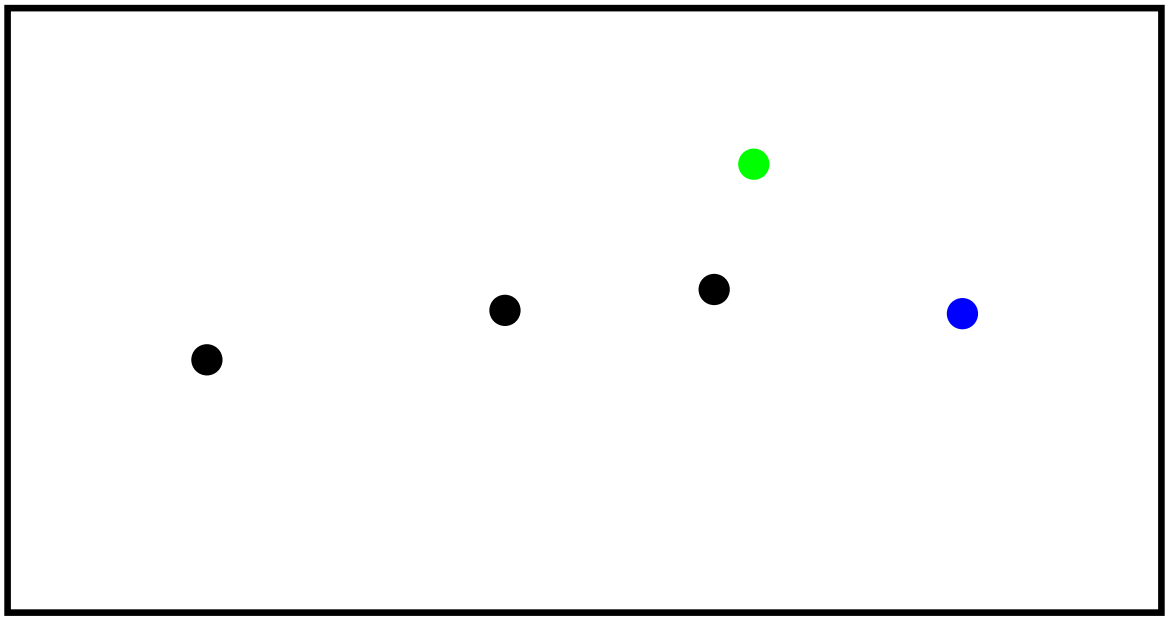


Example 1: starting with 3 blues, 2red



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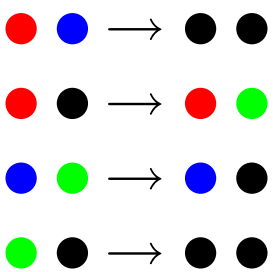

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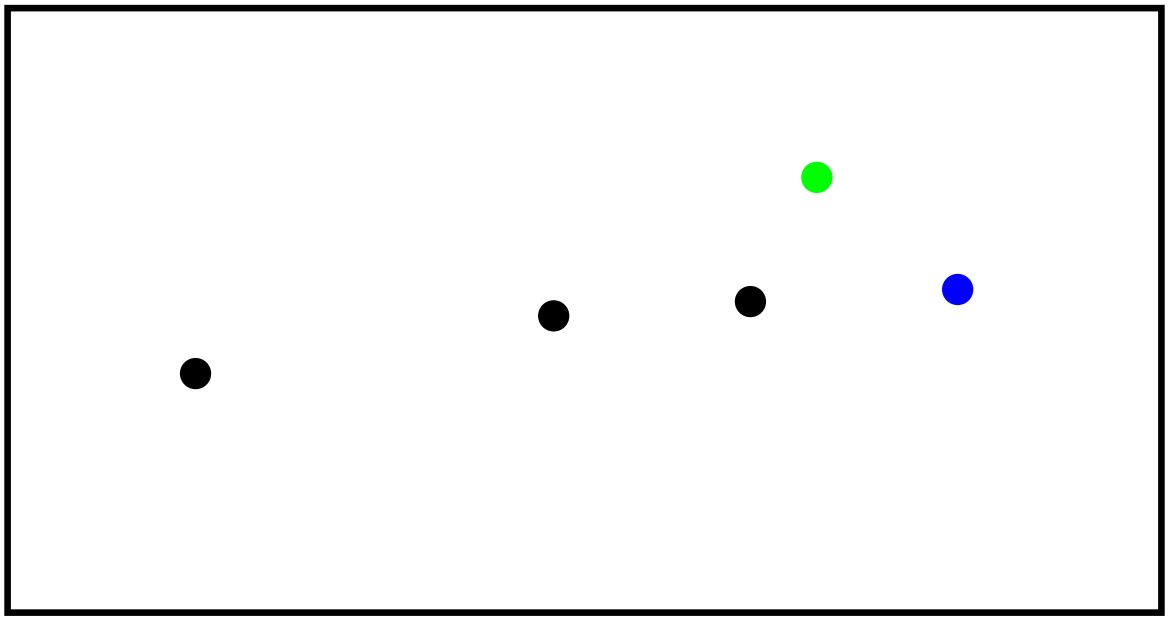

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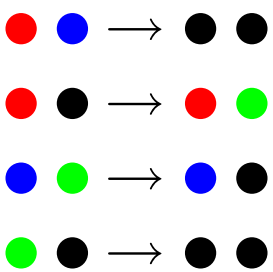

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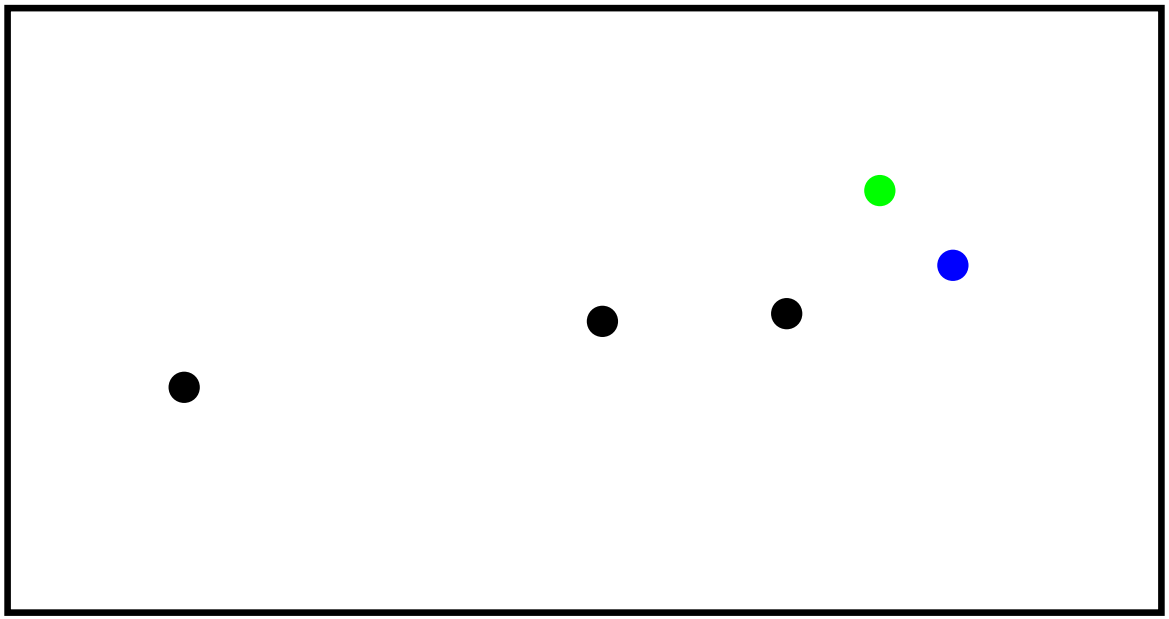

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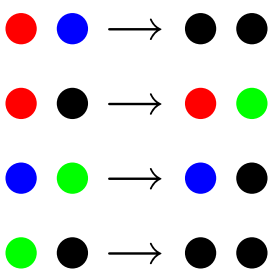

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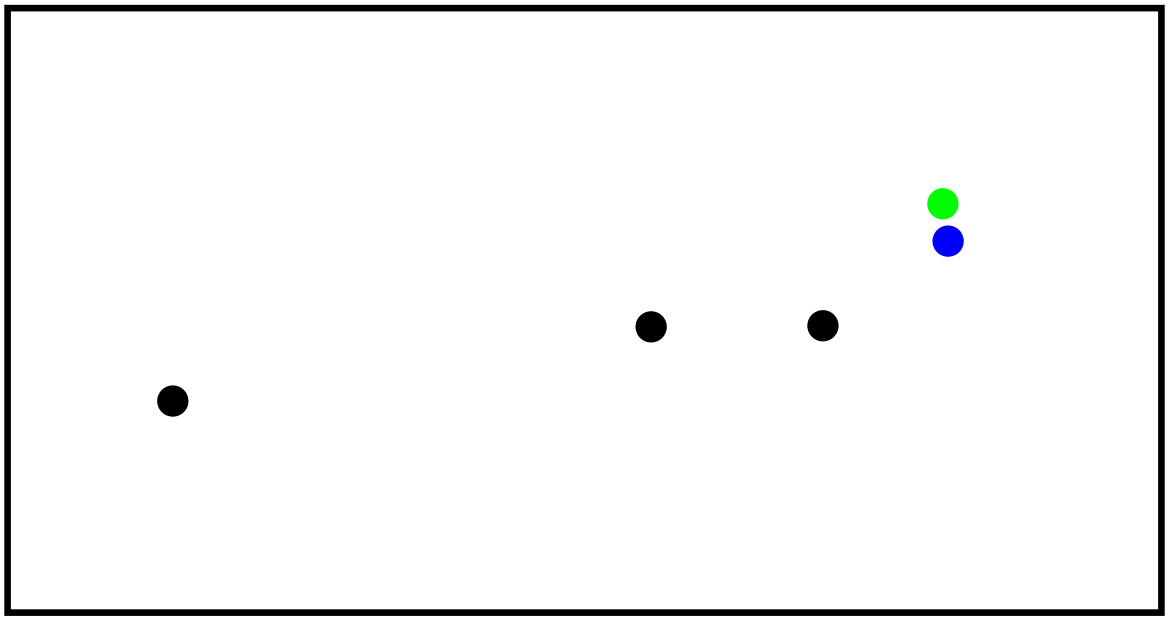

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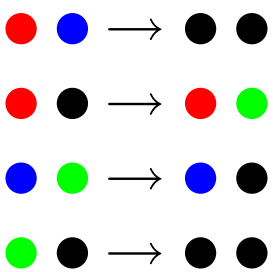

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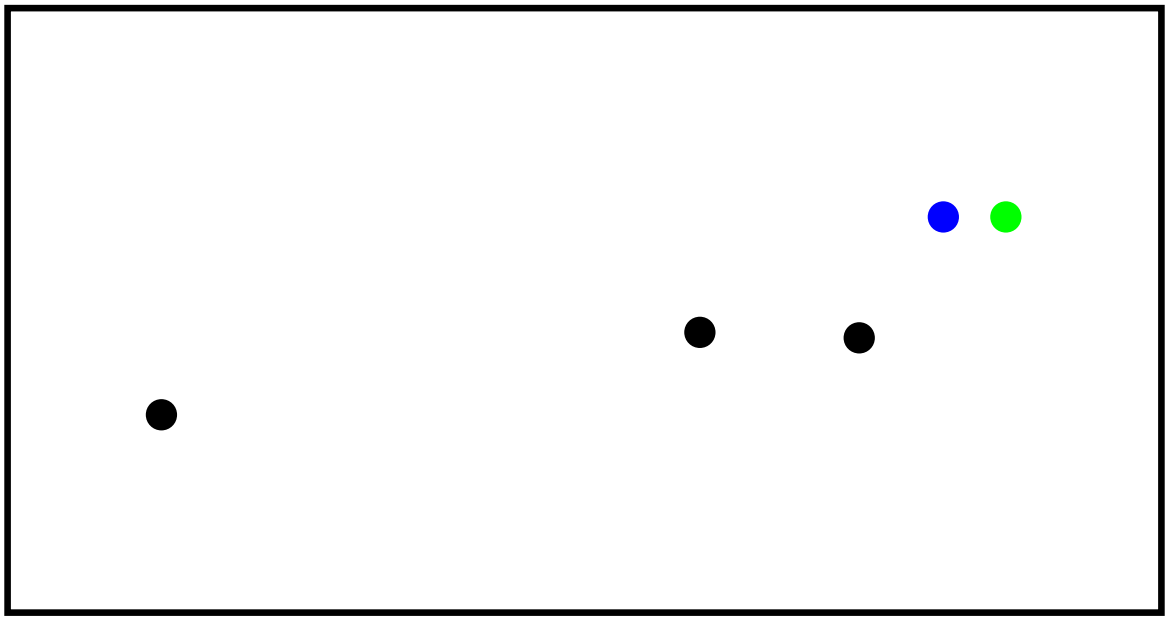

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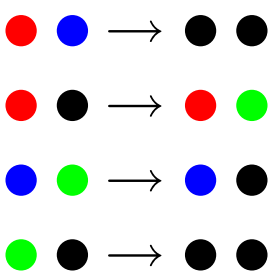
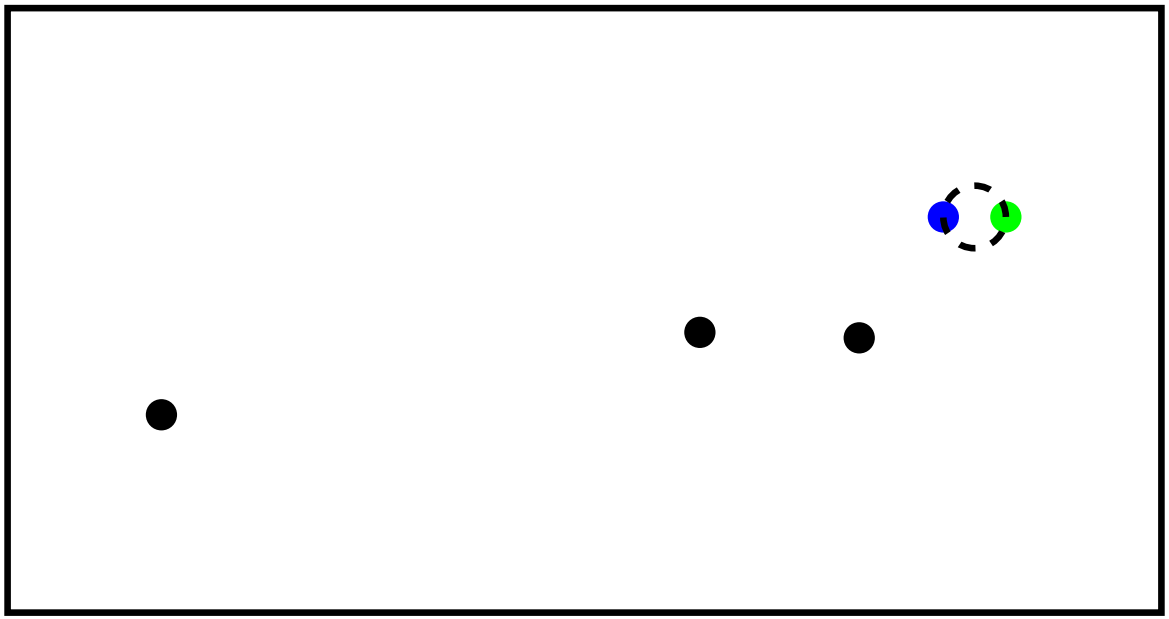


Example 1: starting with 3 blues, 2red



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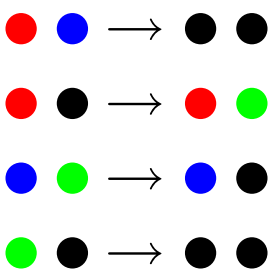

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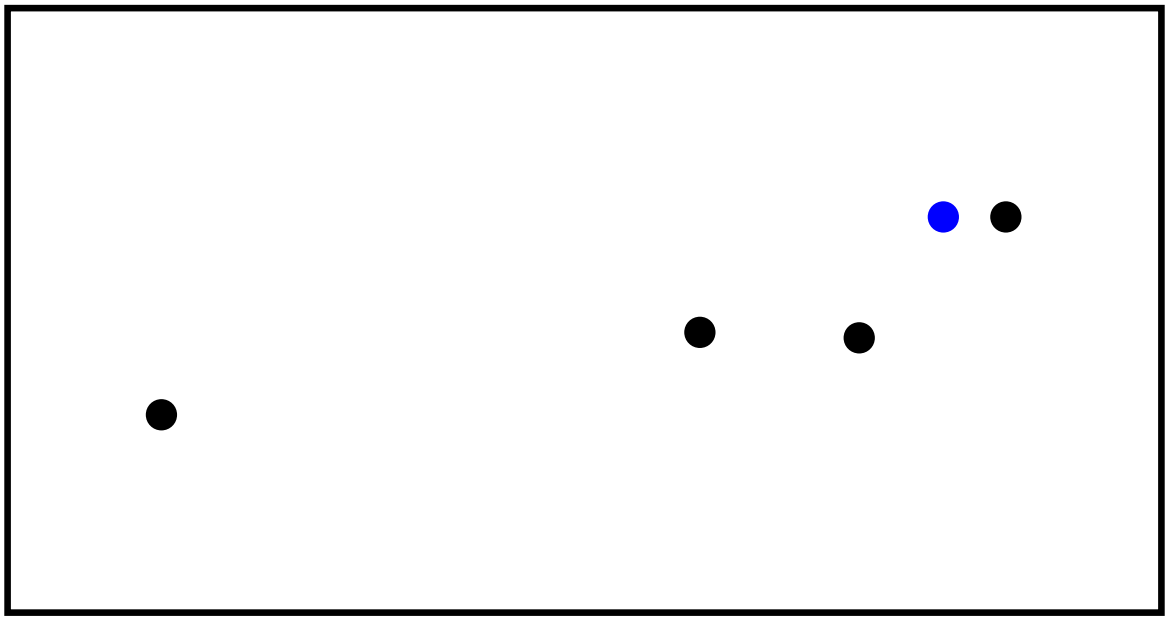

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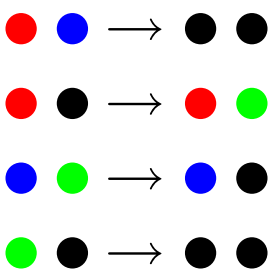
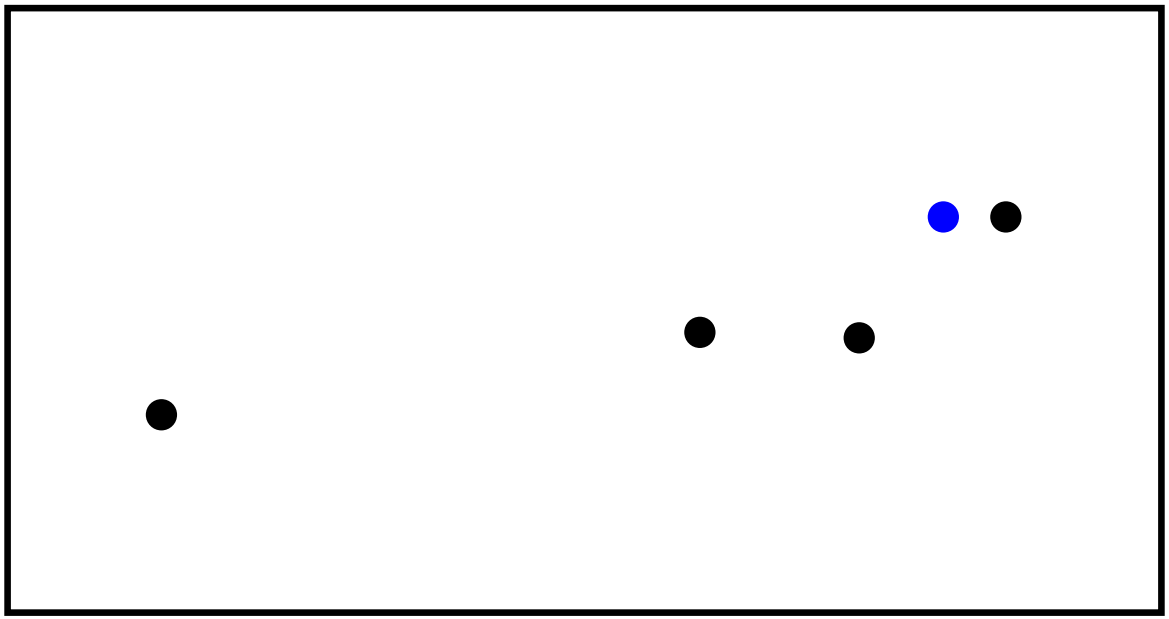


Example 1: Final result



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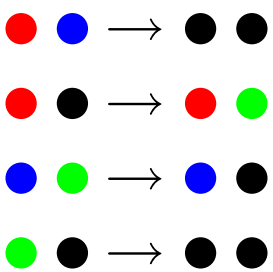
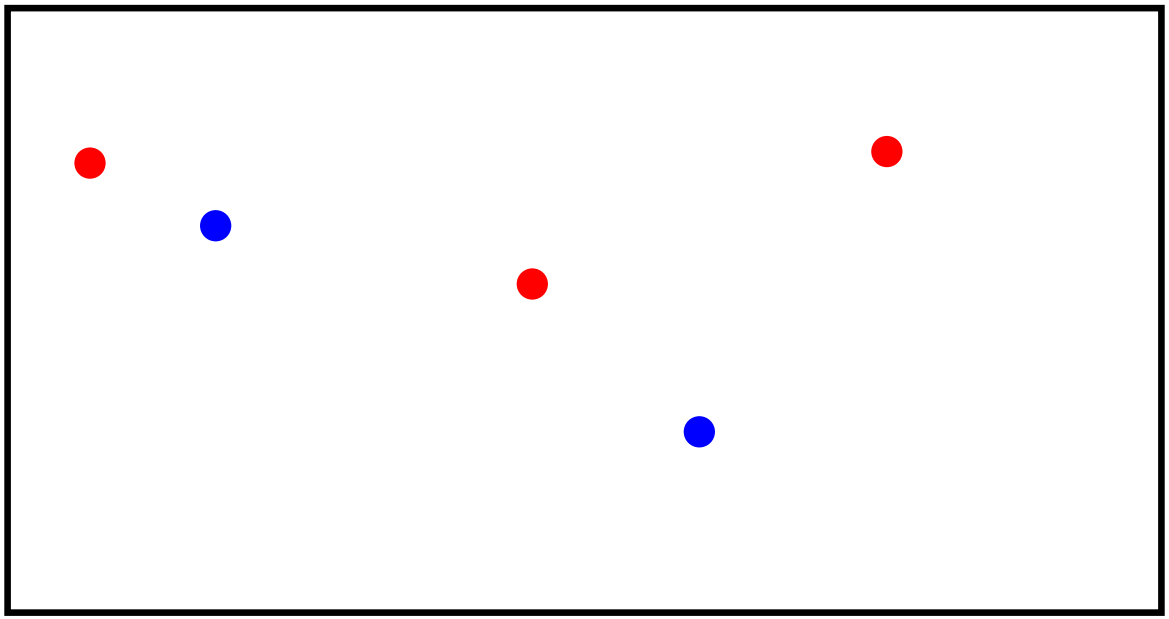


Example 2: starting with 2 blues, 3red



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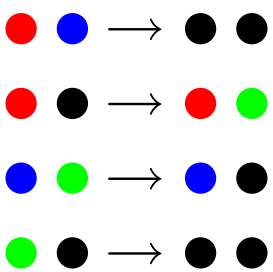

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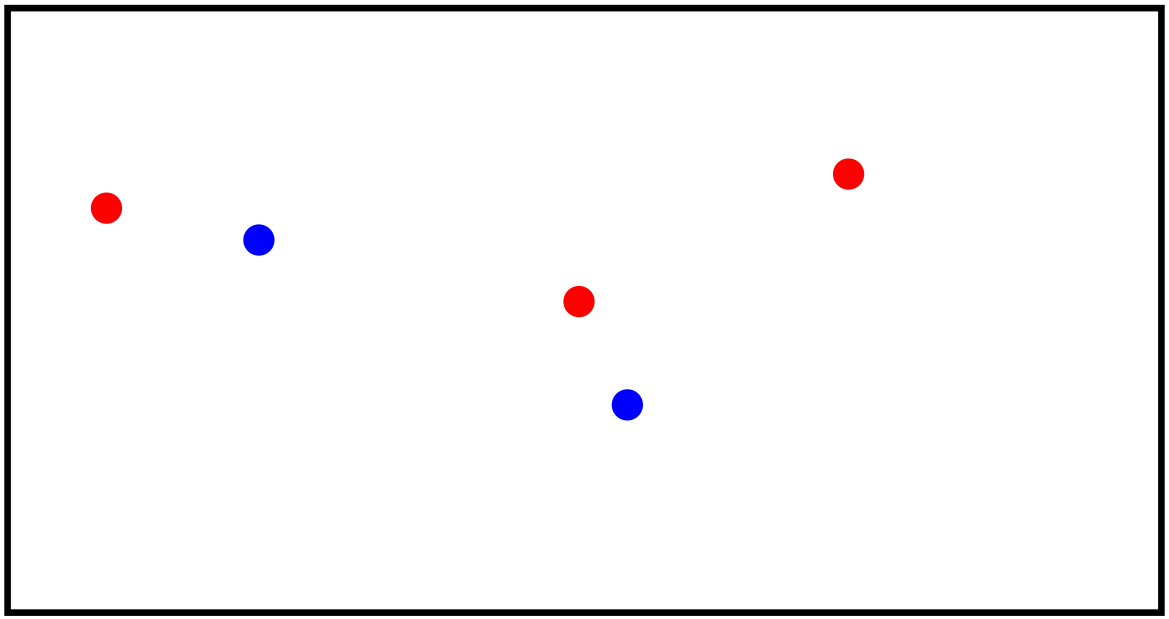

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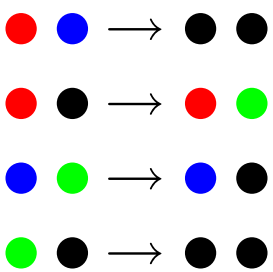

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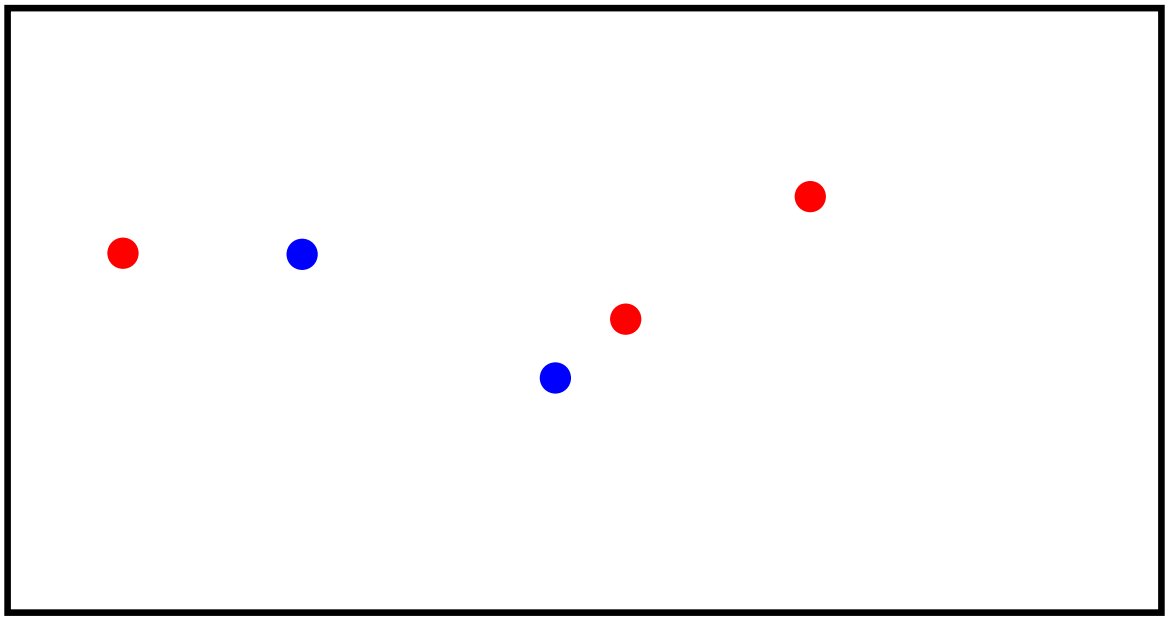

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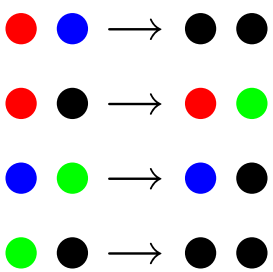

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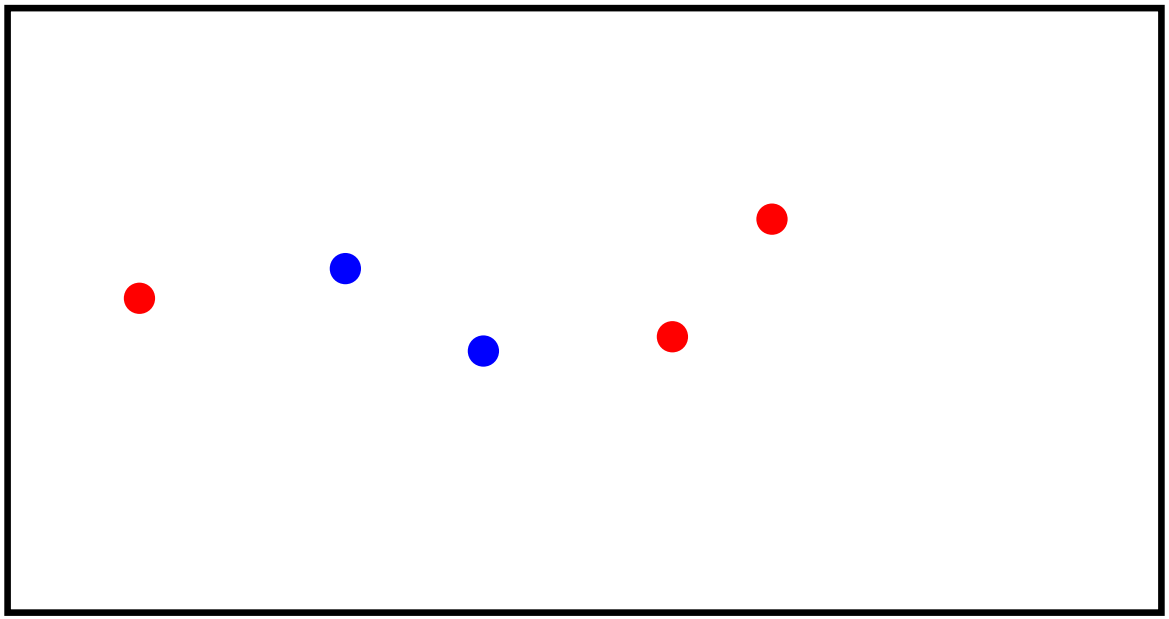

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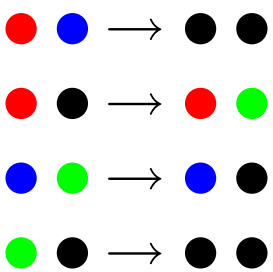

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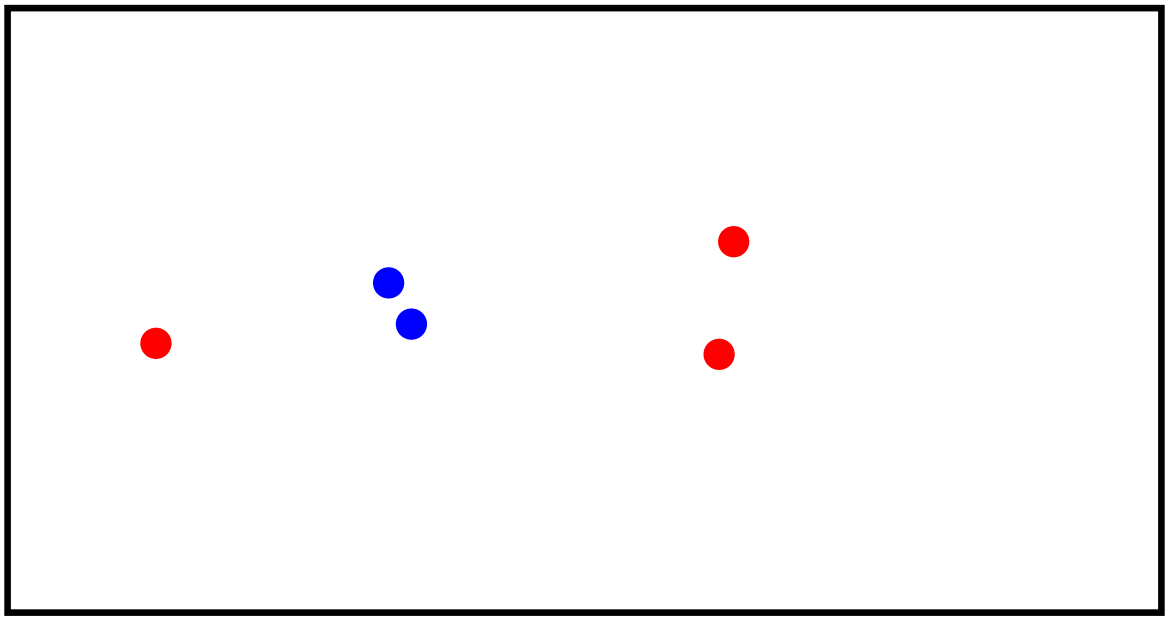

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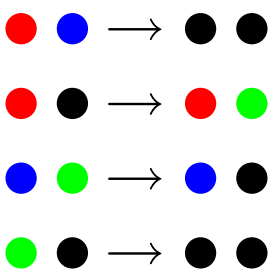

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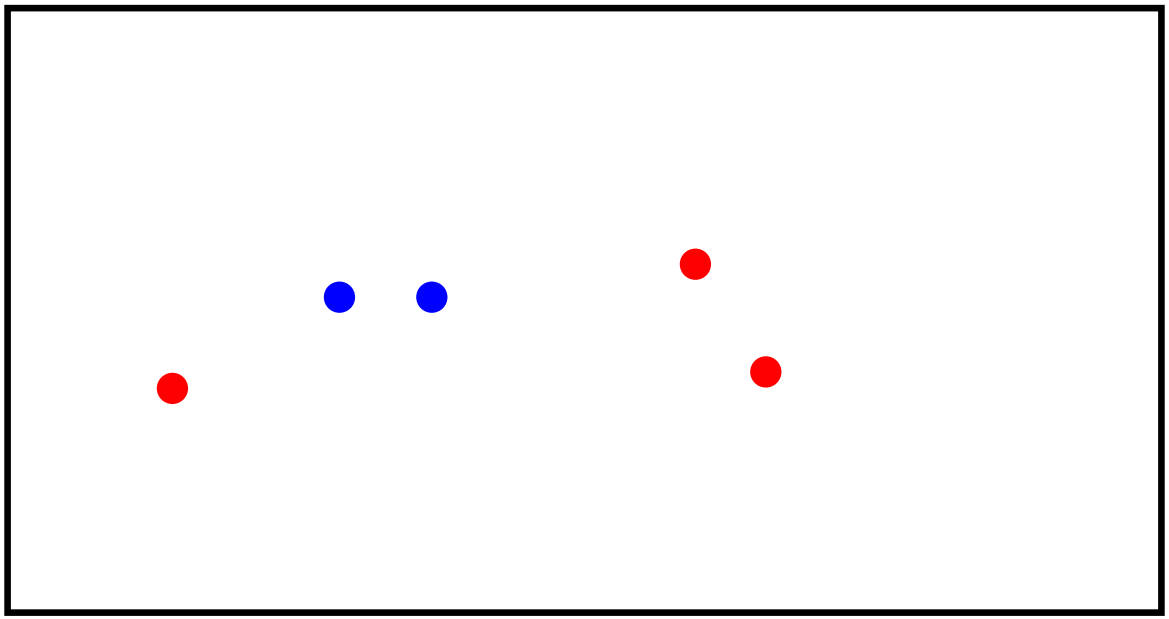

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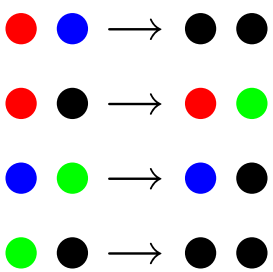

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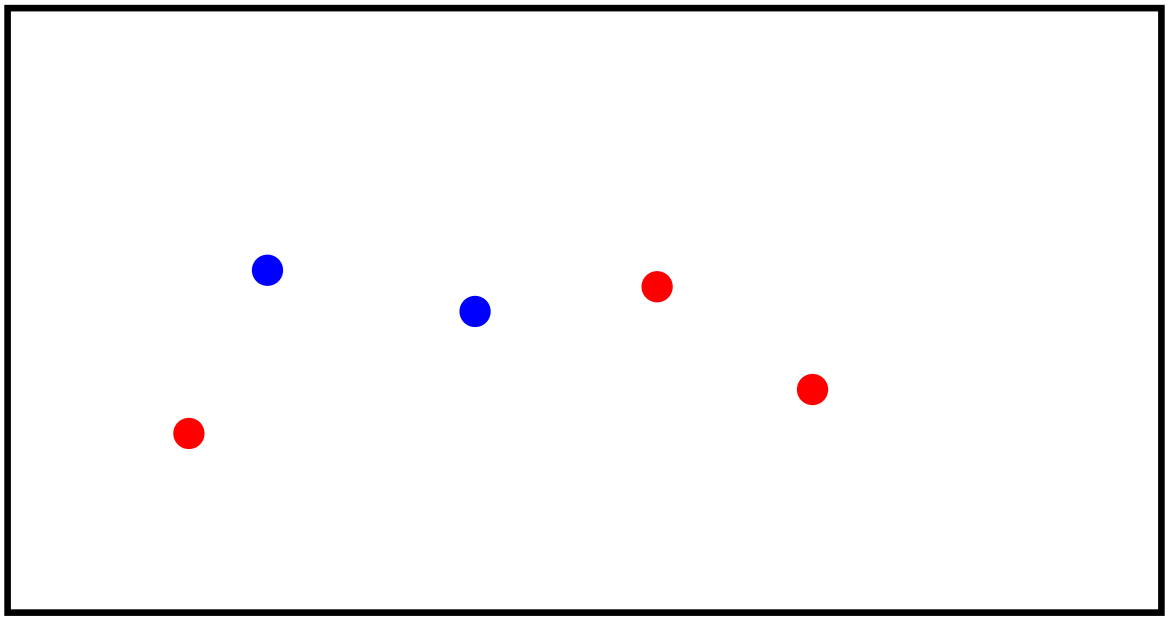

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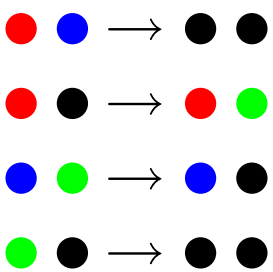

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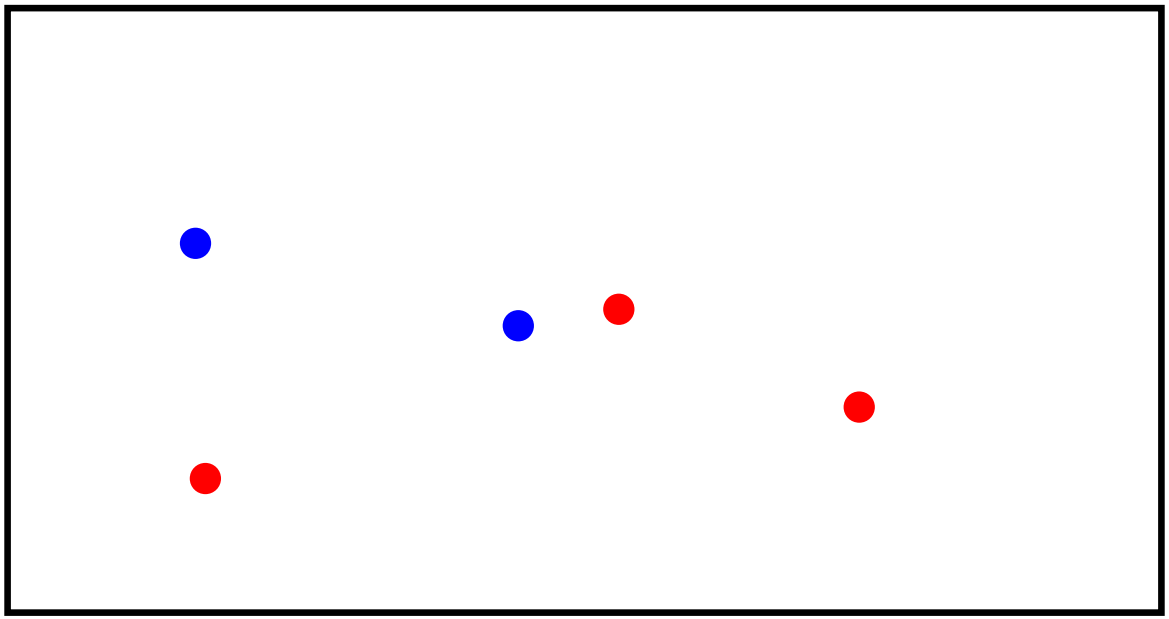

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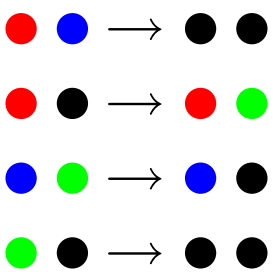

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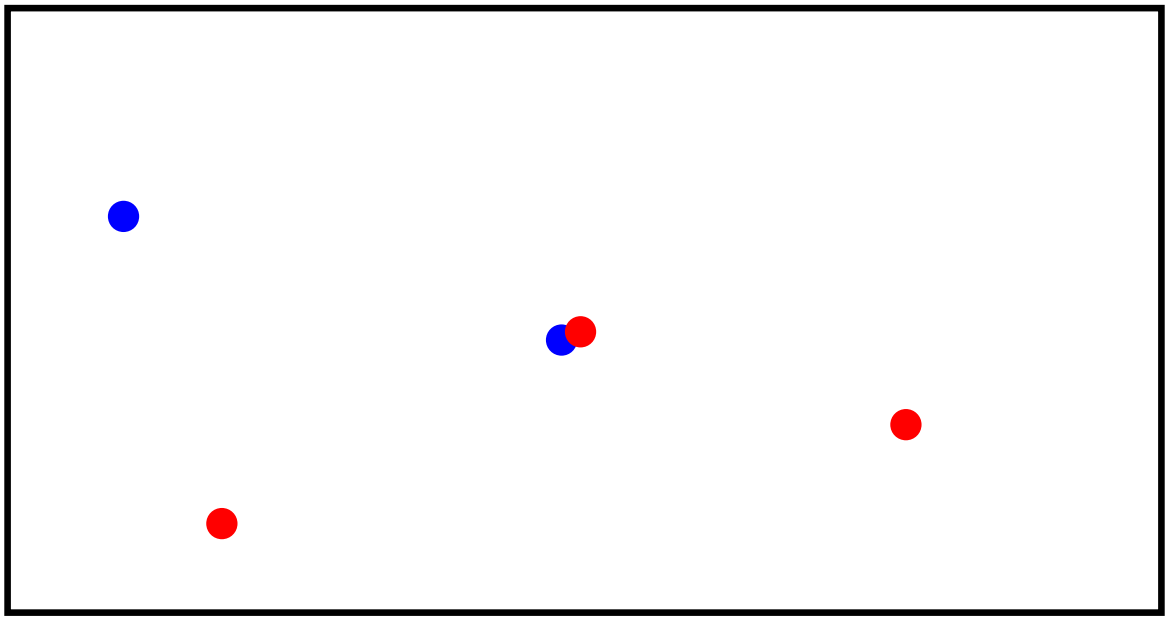

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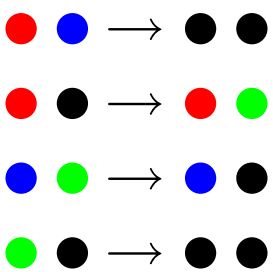
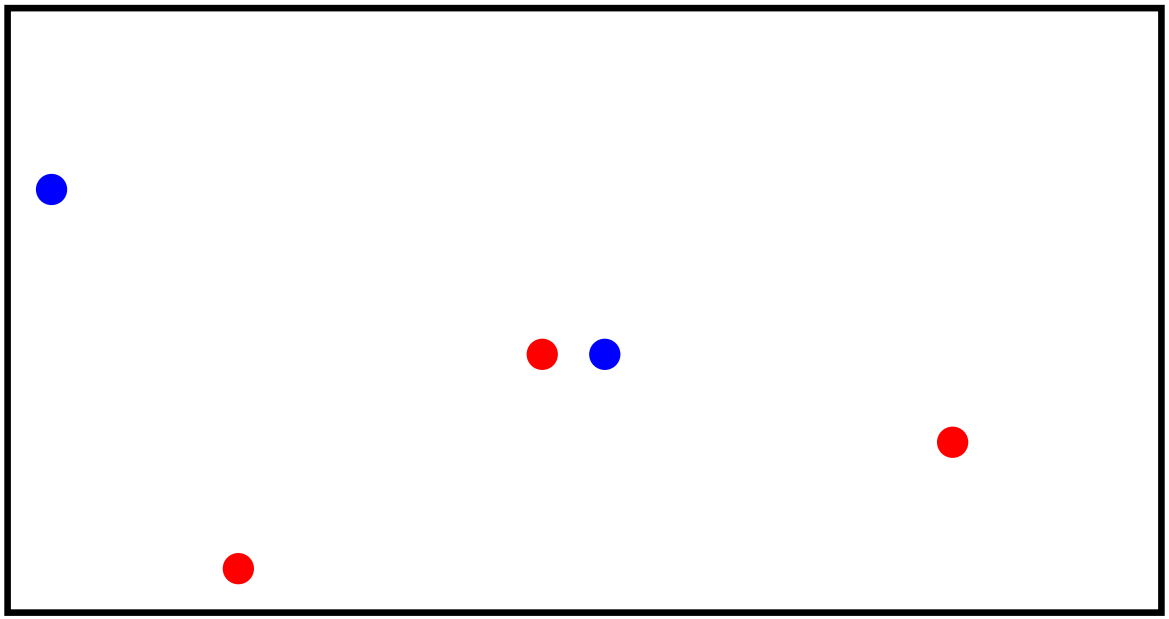


Example 2: starting with 2 blues, 3red



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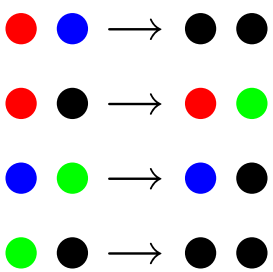

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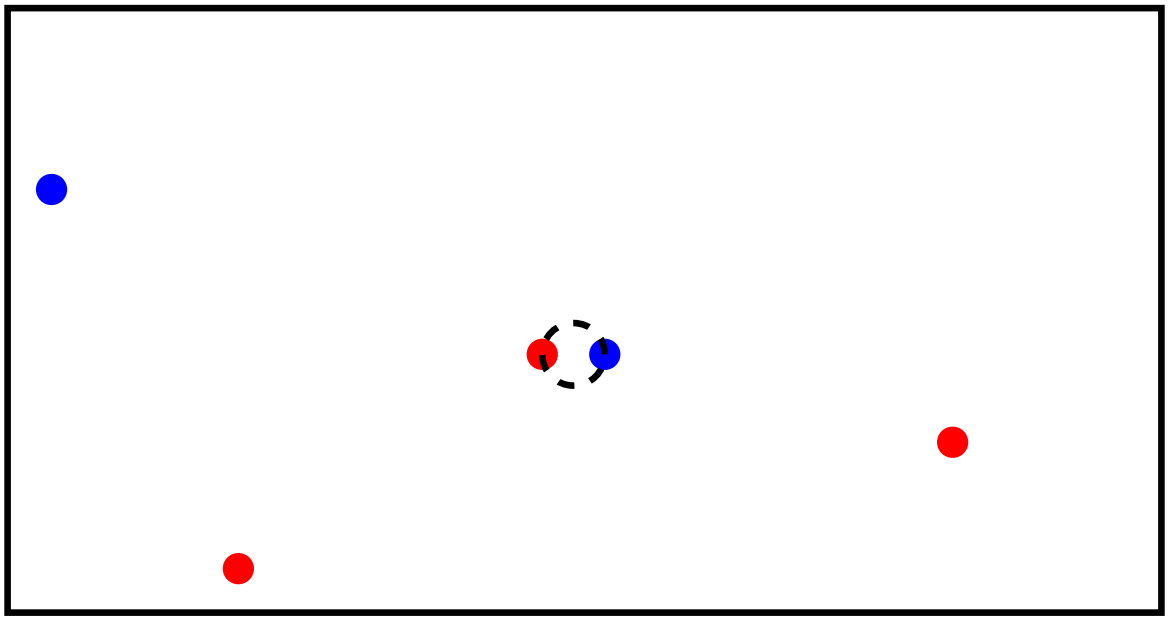

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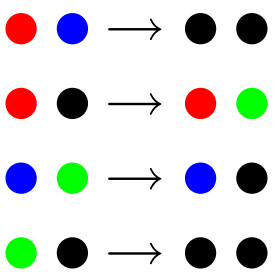

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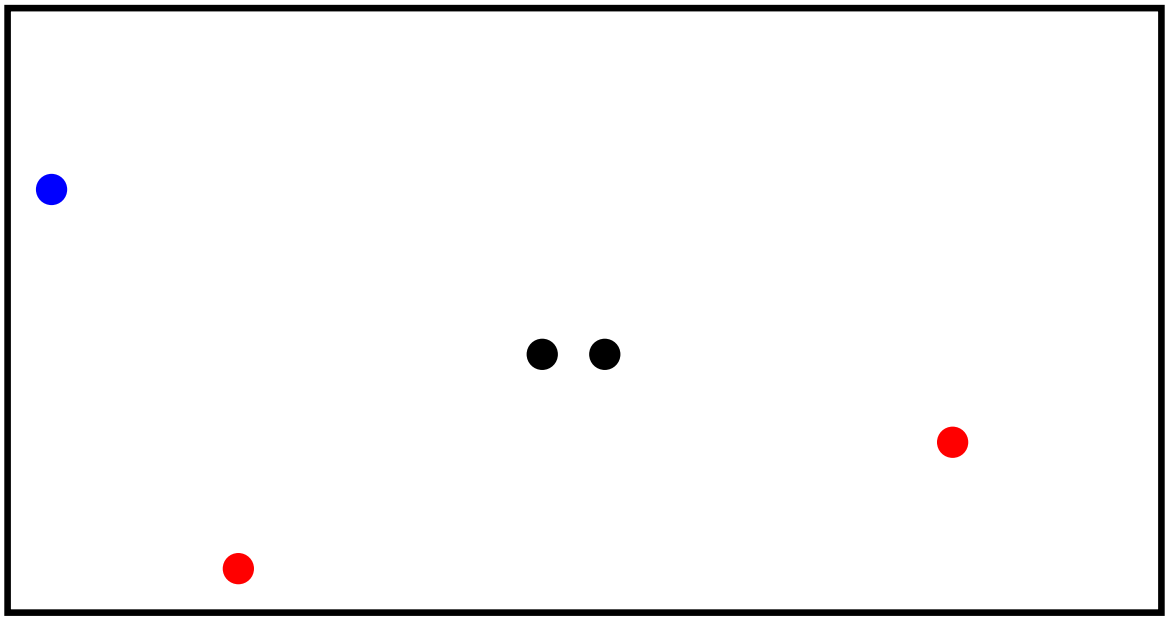

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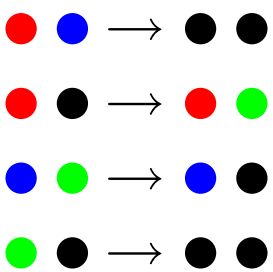

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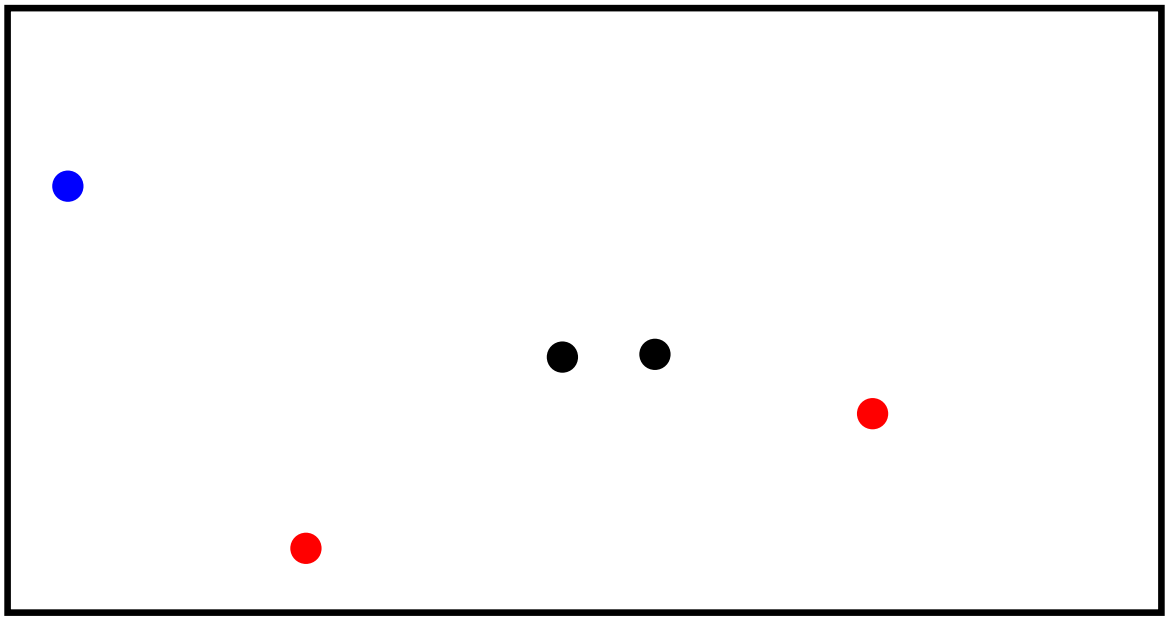

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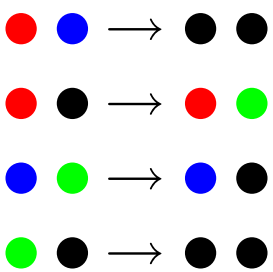

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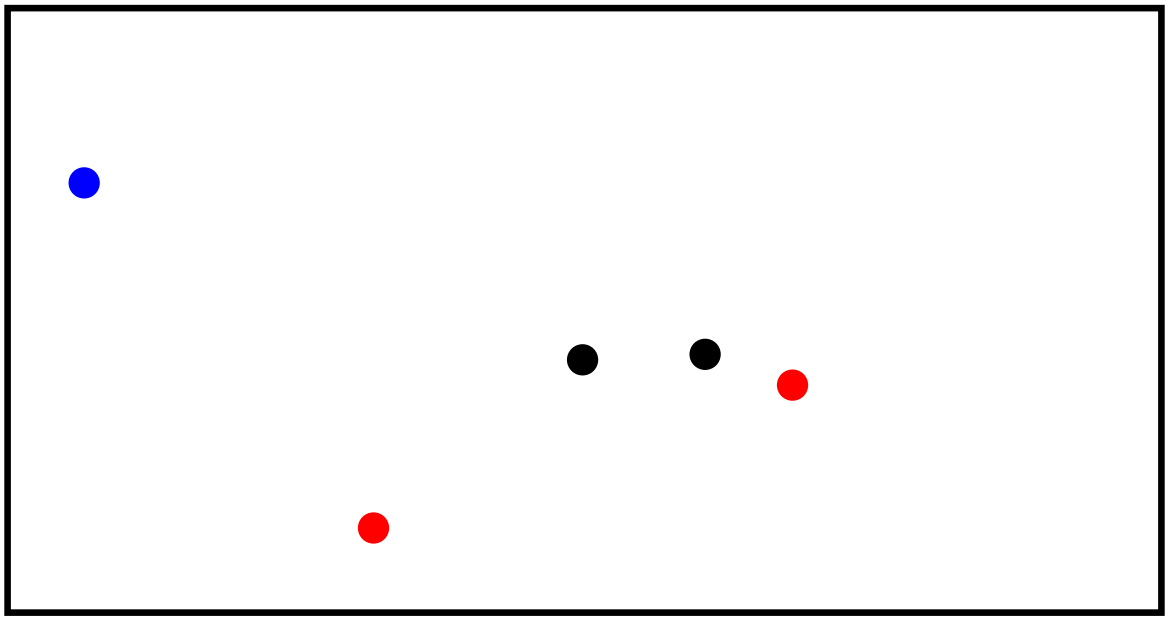

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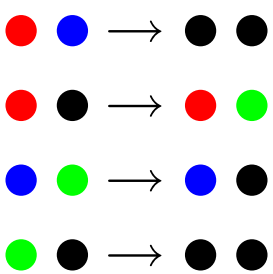

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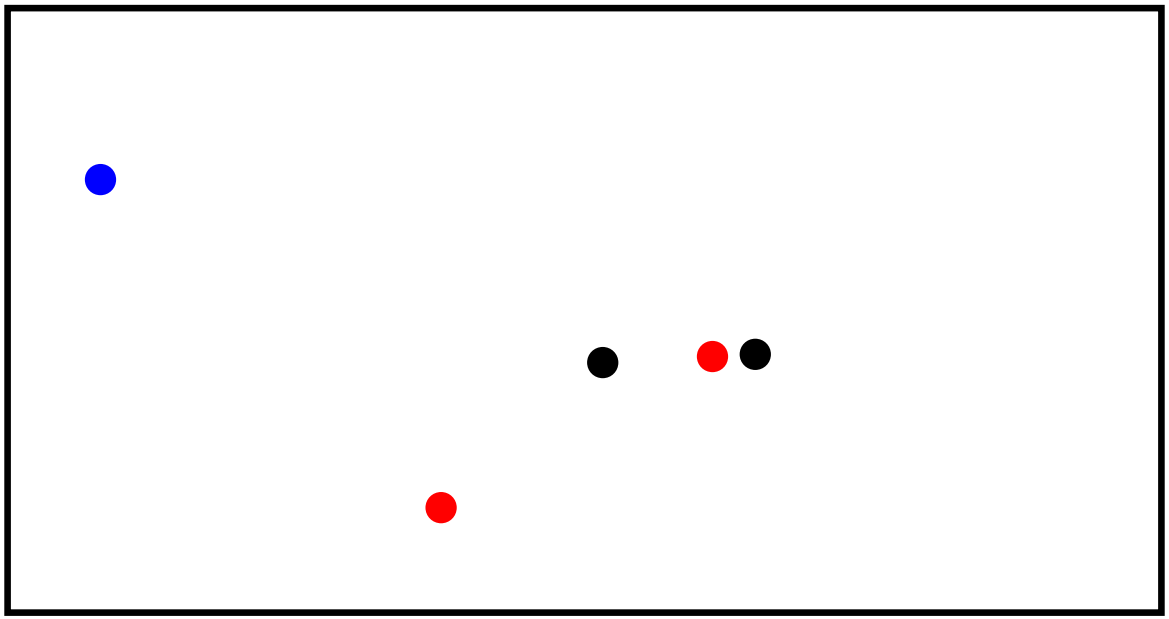

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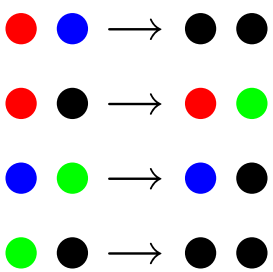

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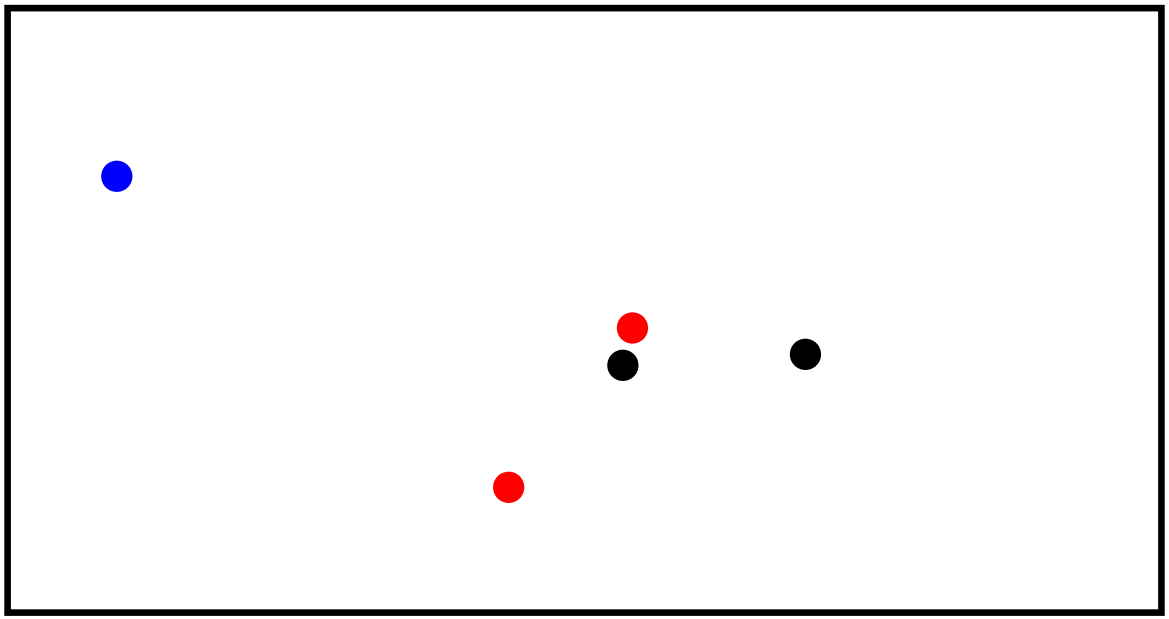

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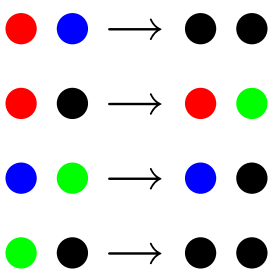

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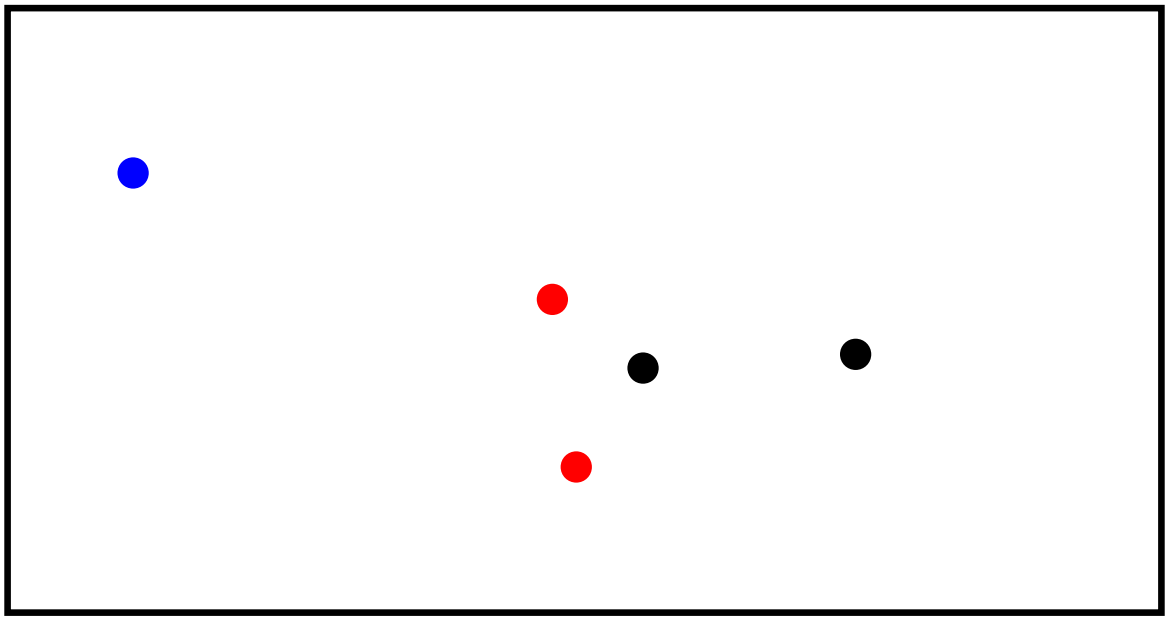

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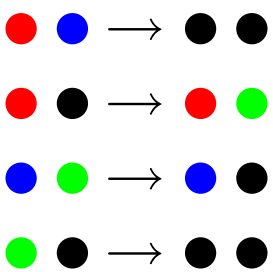

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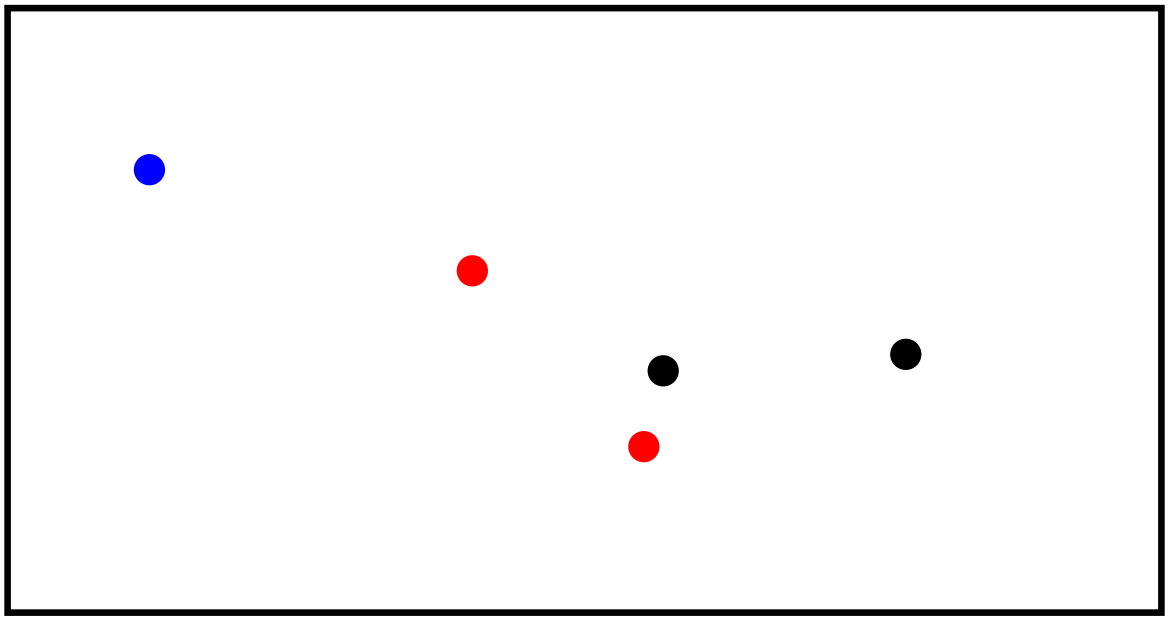

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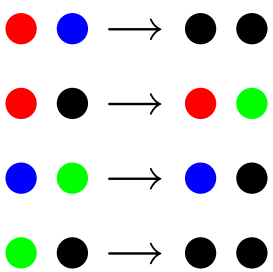

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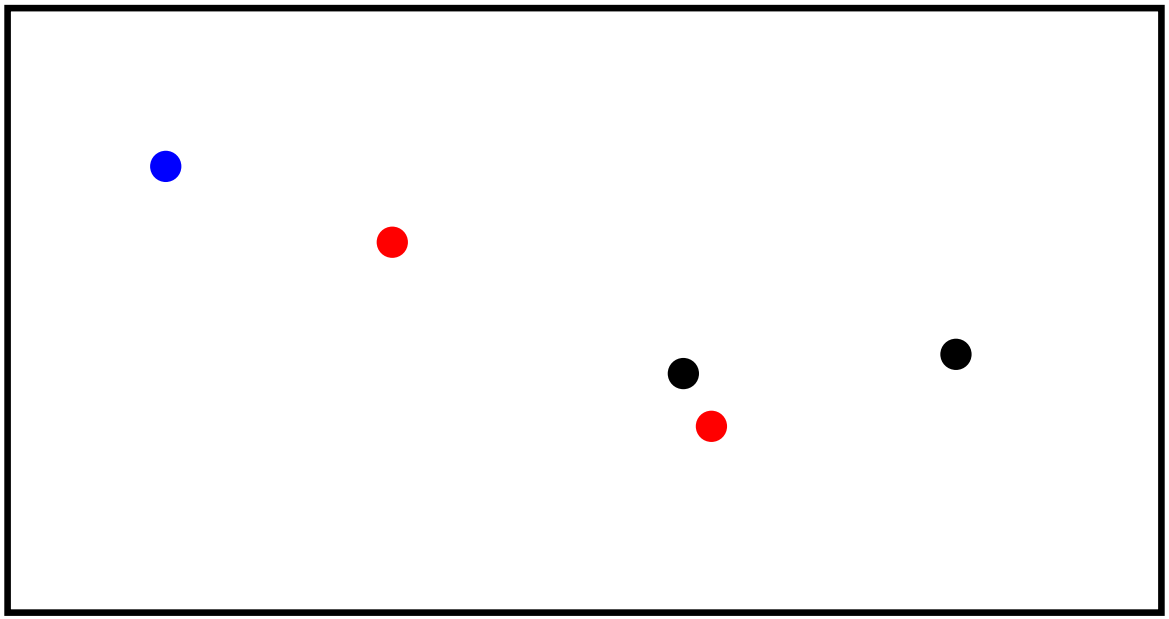

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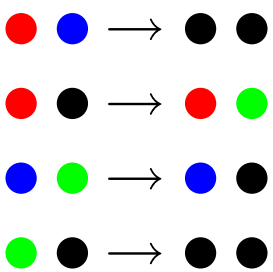

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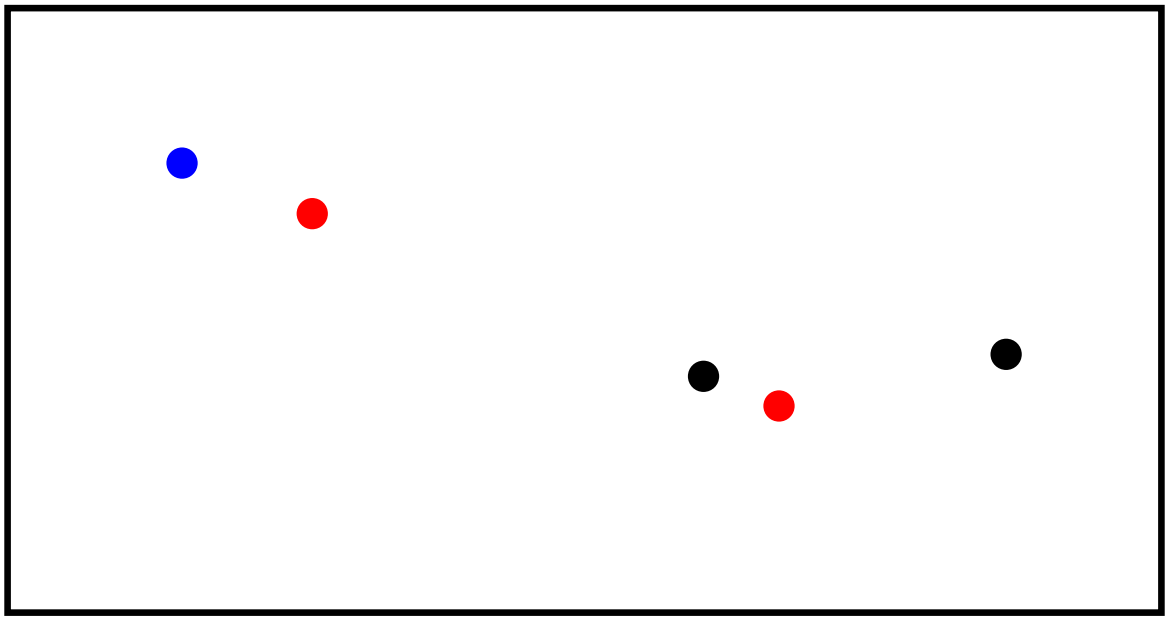

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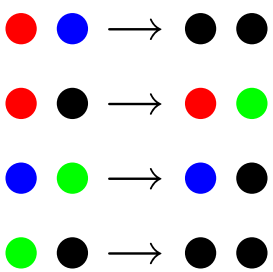

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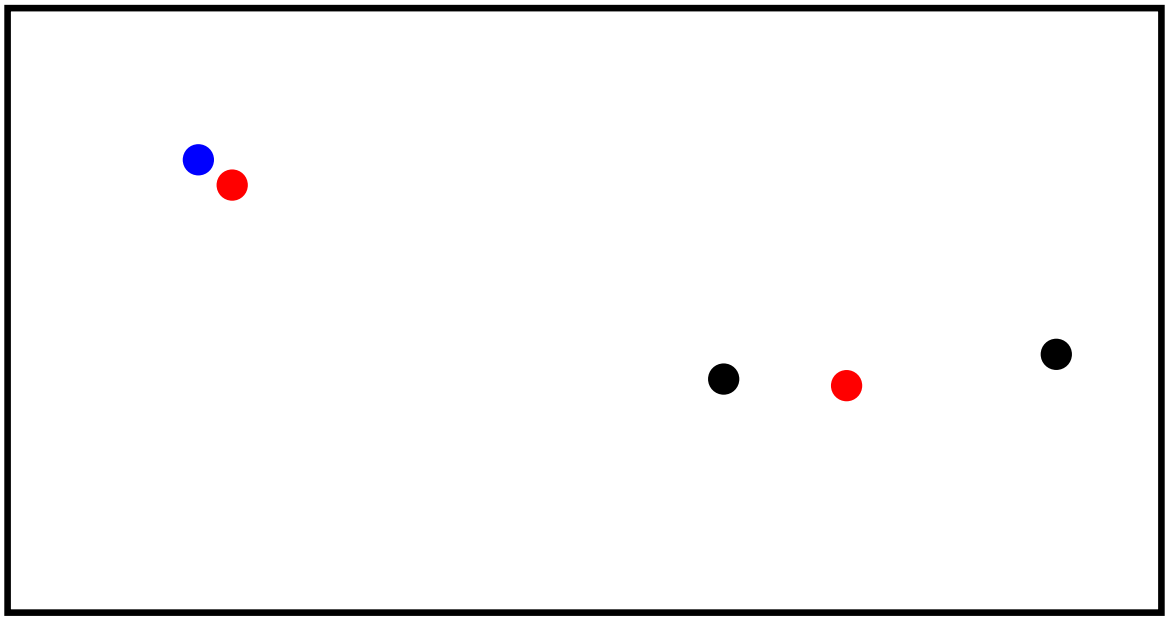

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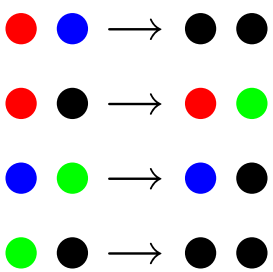

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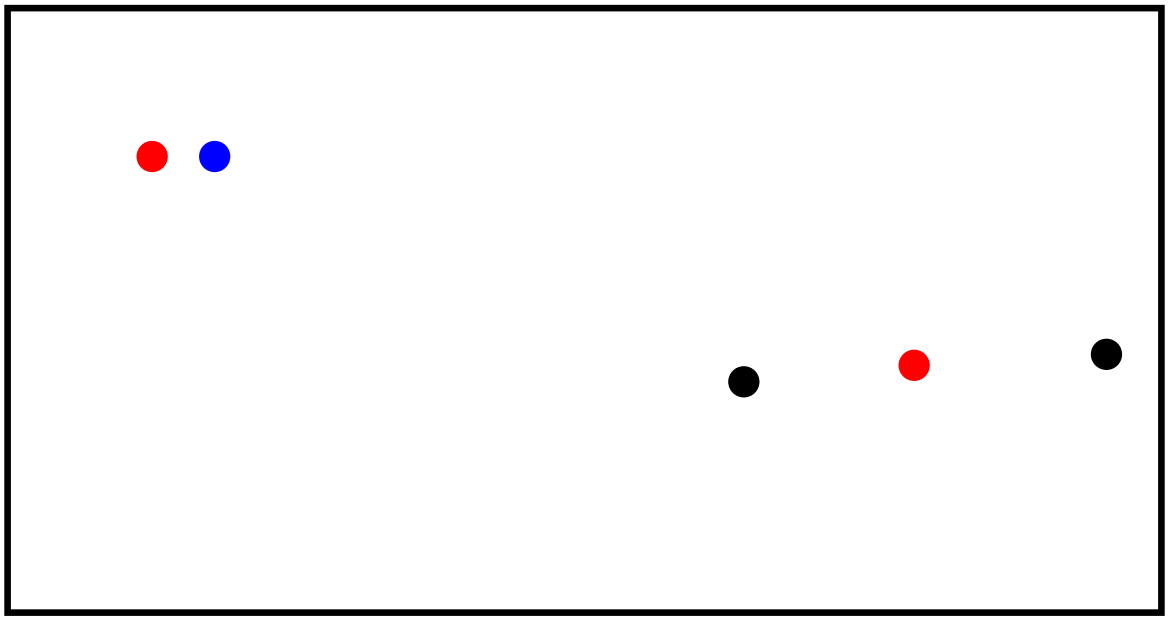

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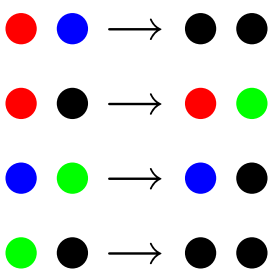

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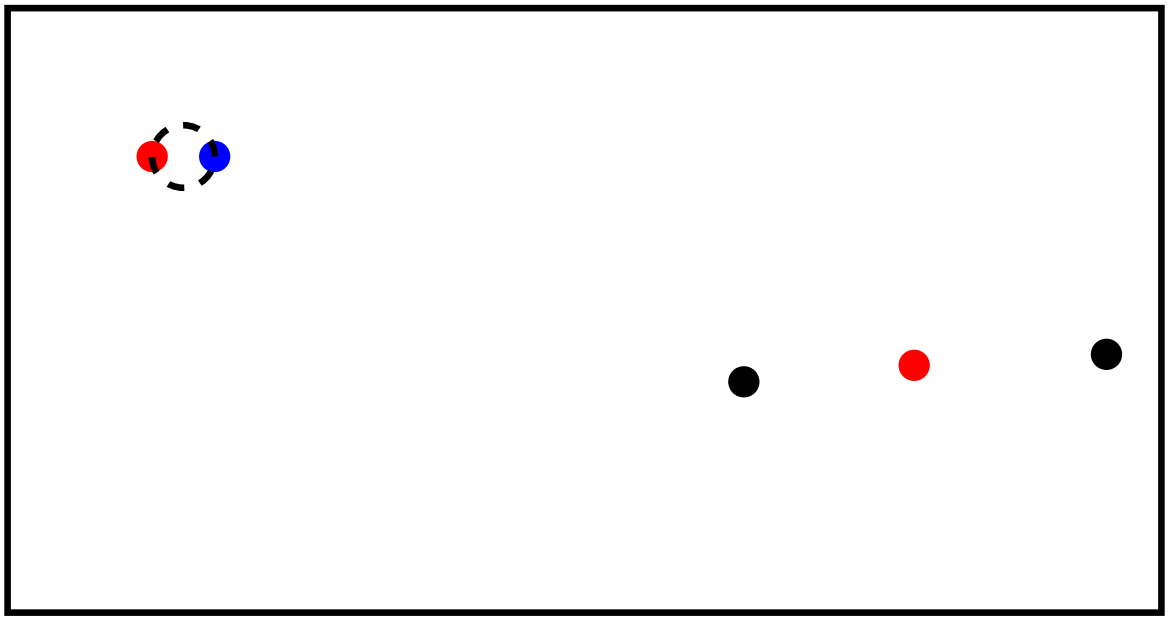

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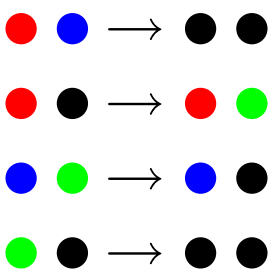

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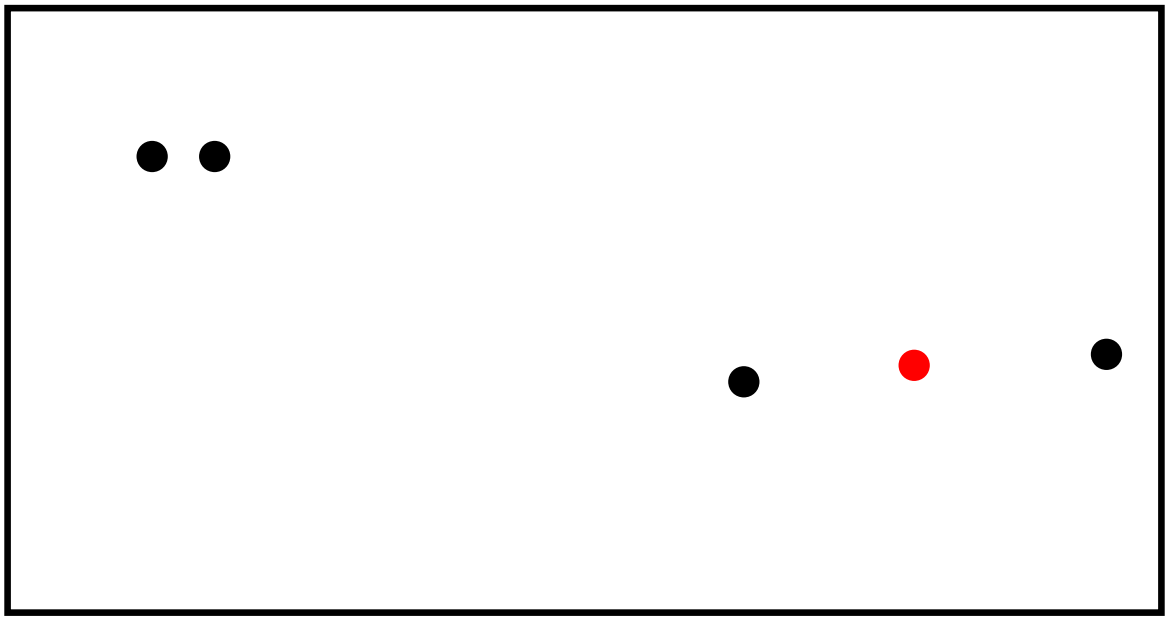

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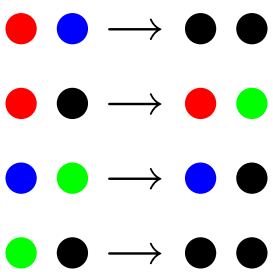

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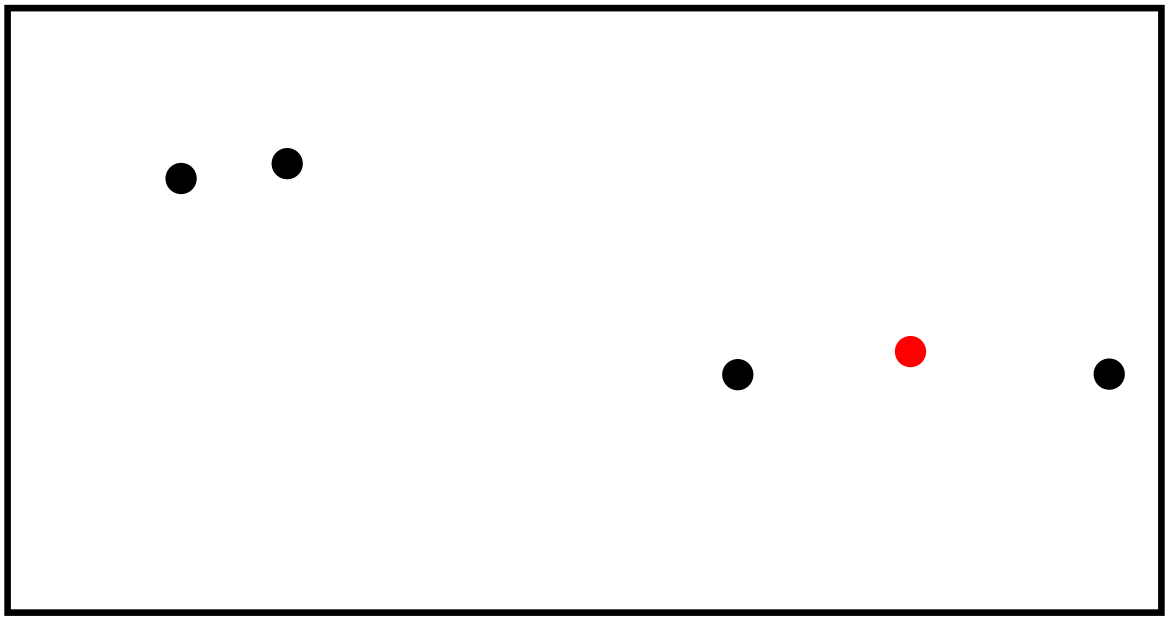

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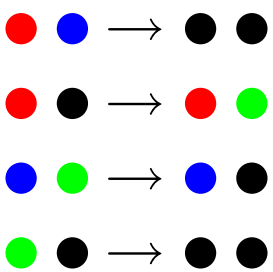

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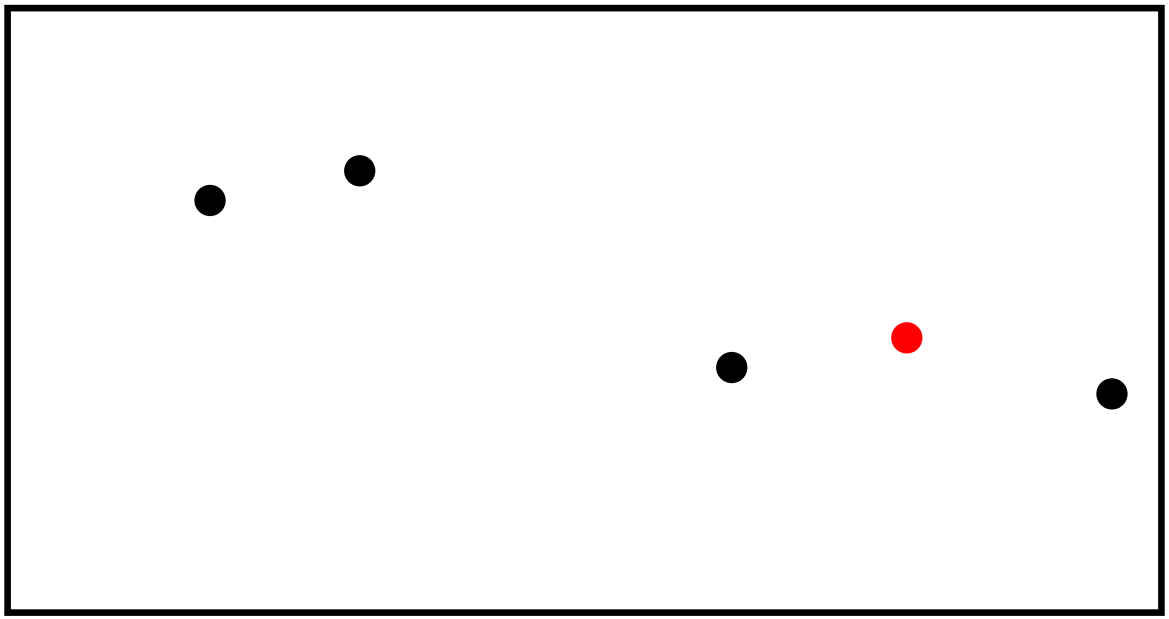

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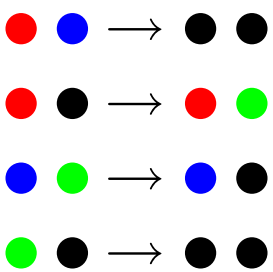

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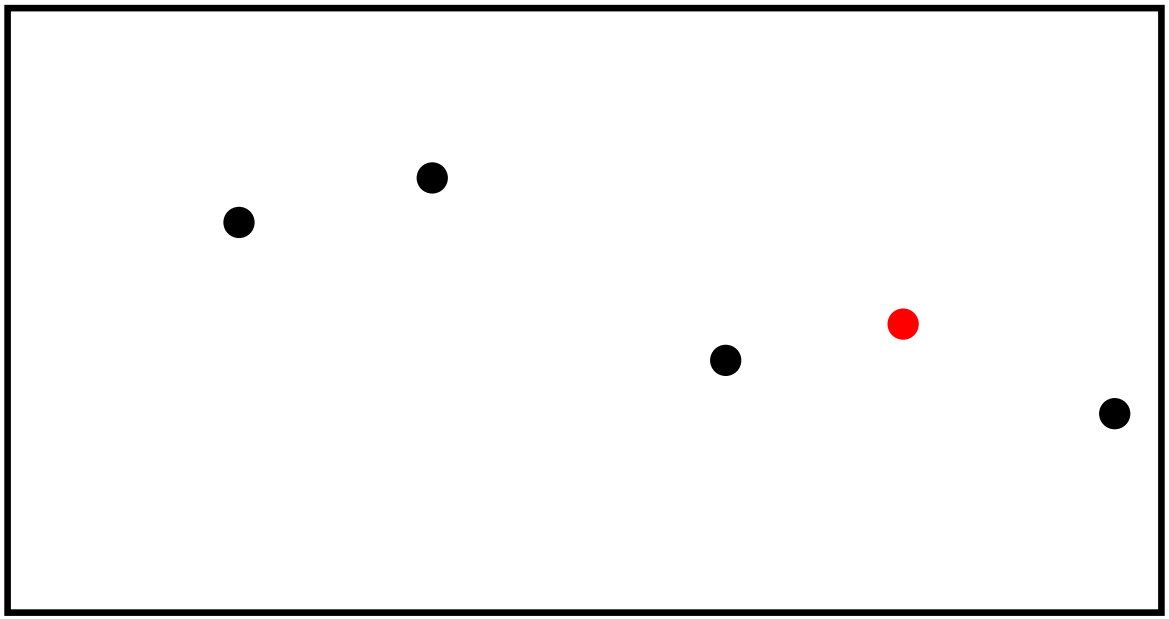

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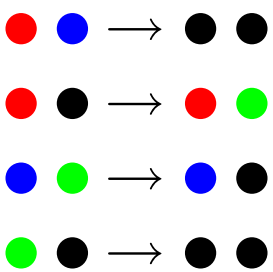

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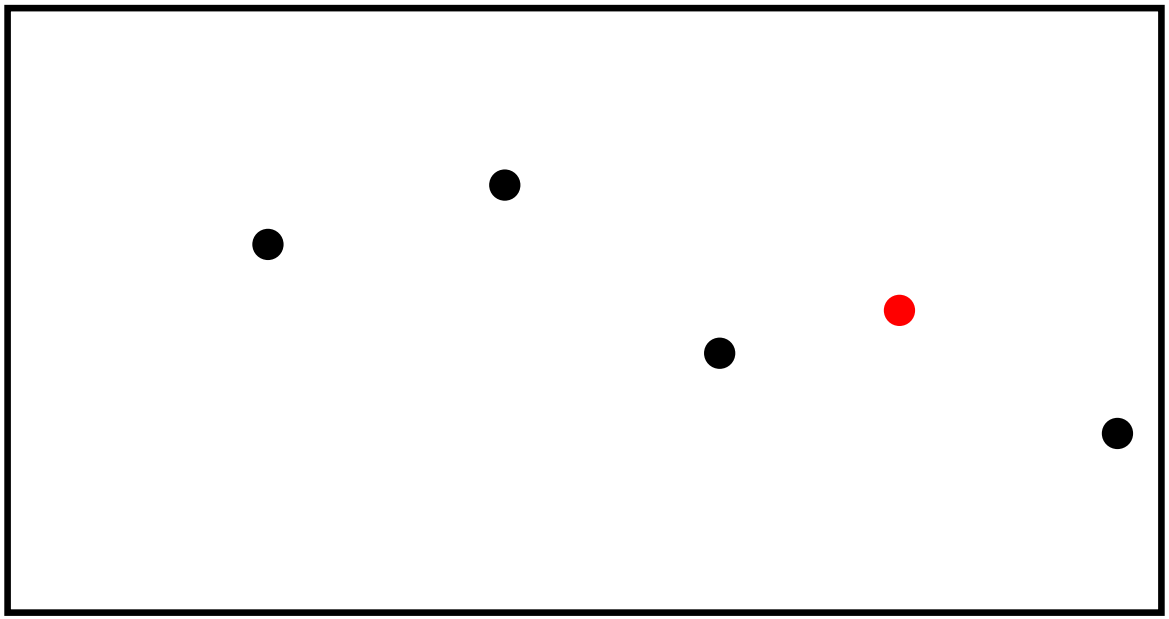

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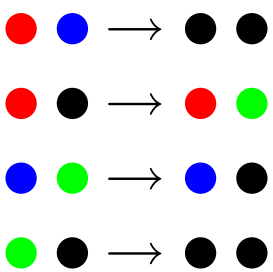

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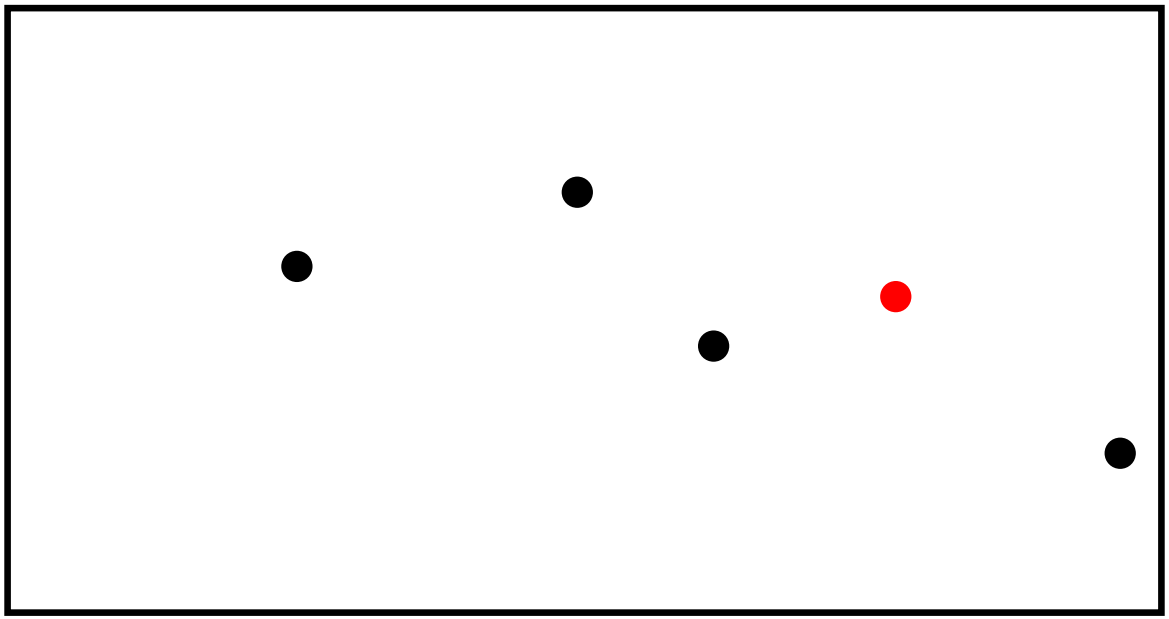

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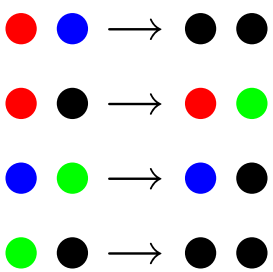

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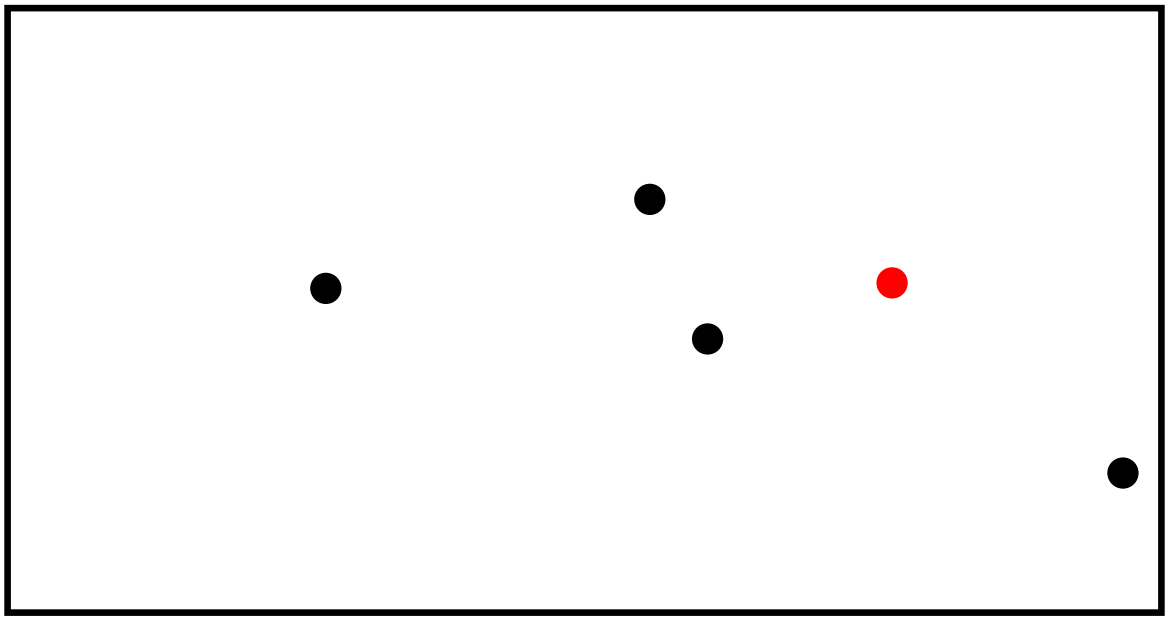

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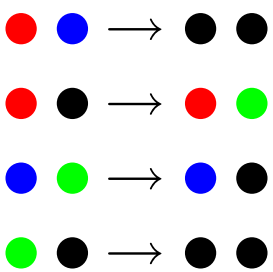

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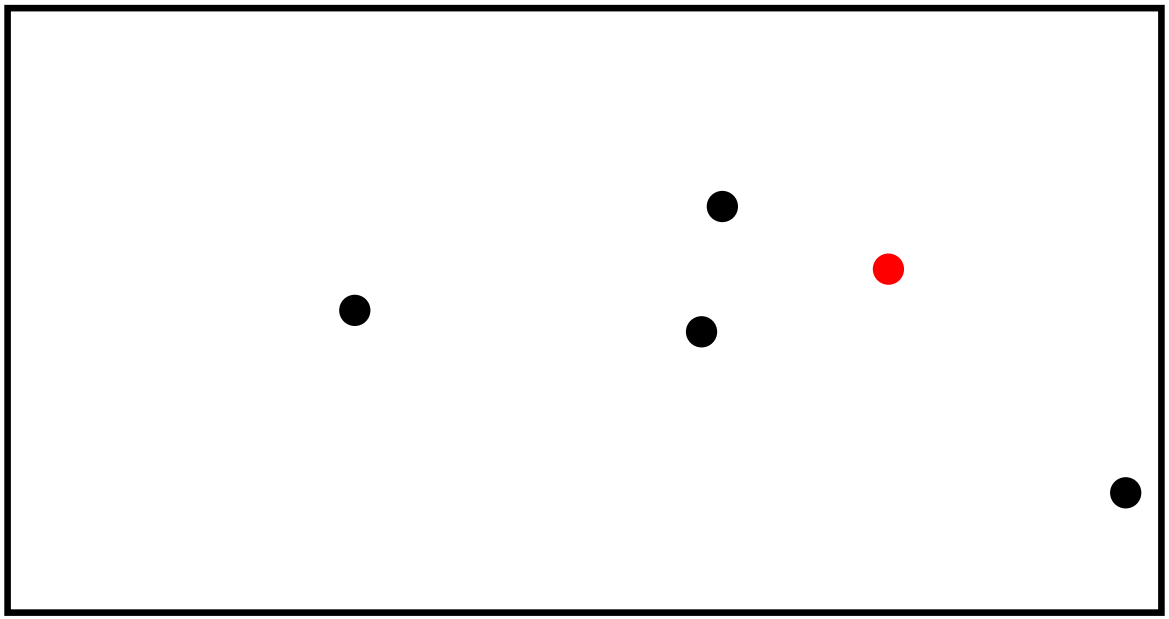

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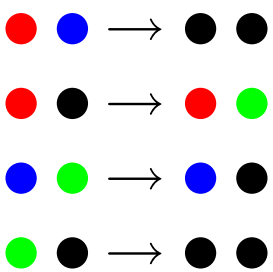

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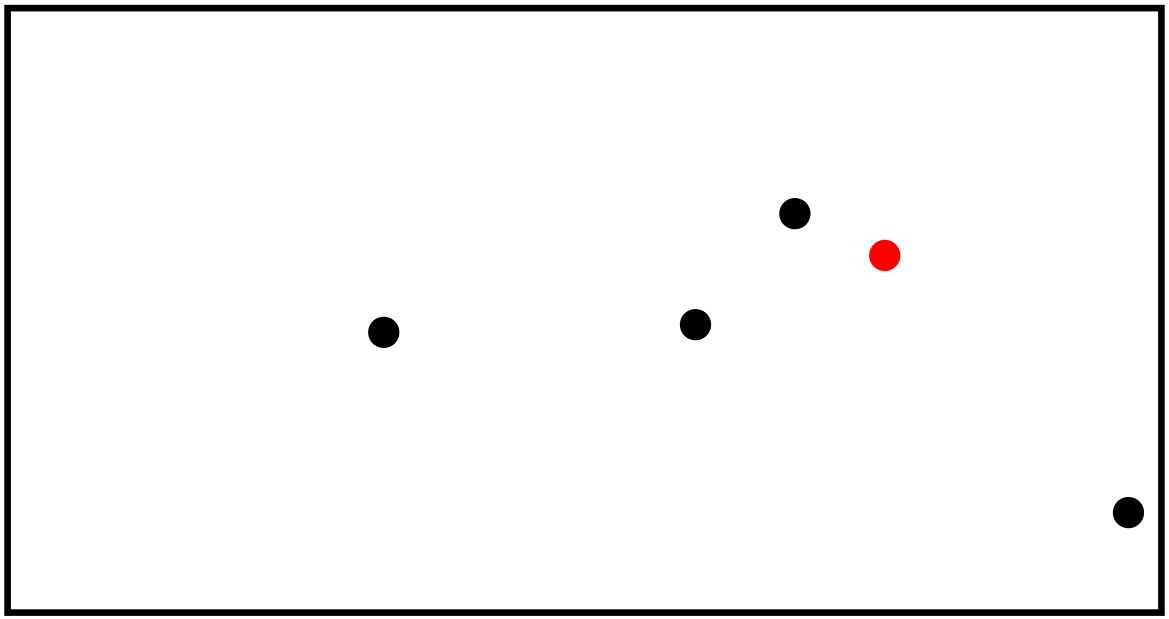

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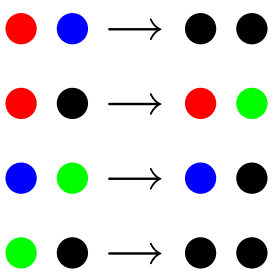

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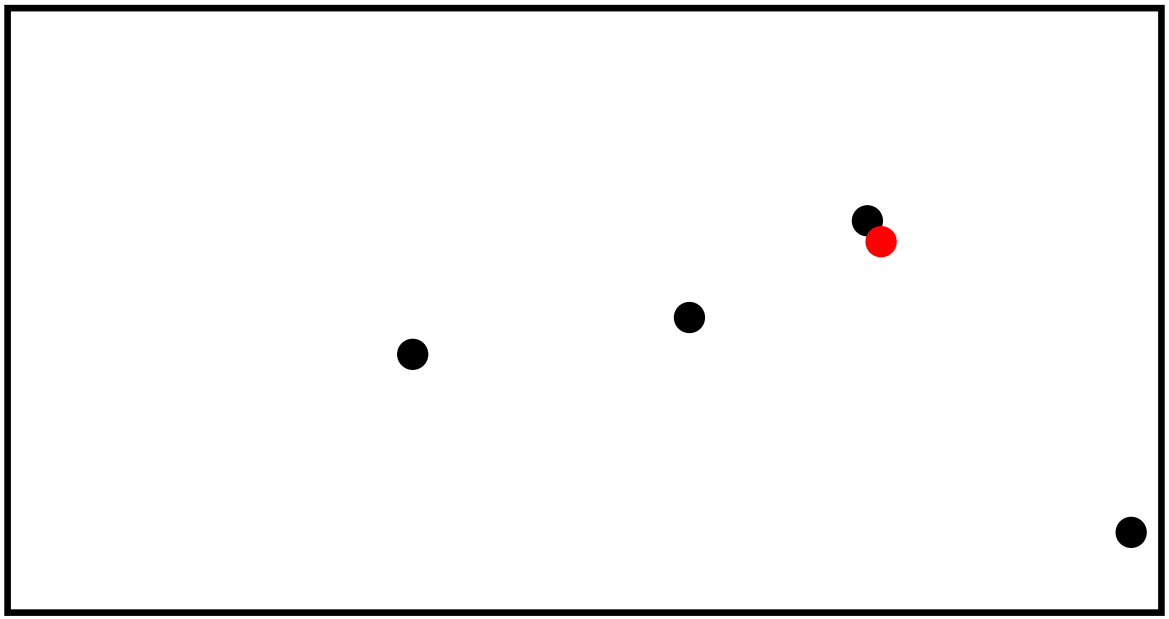

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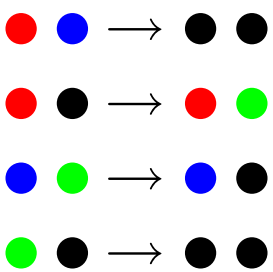

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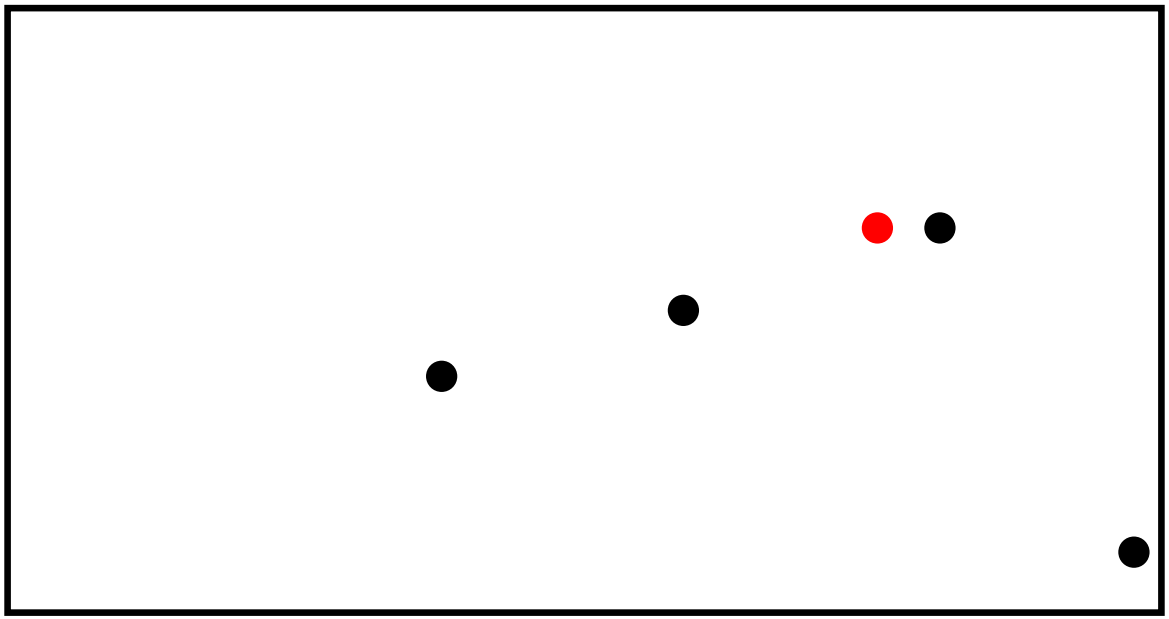

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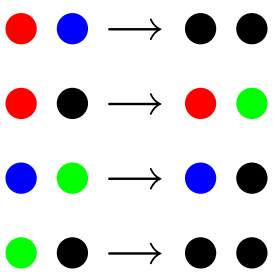

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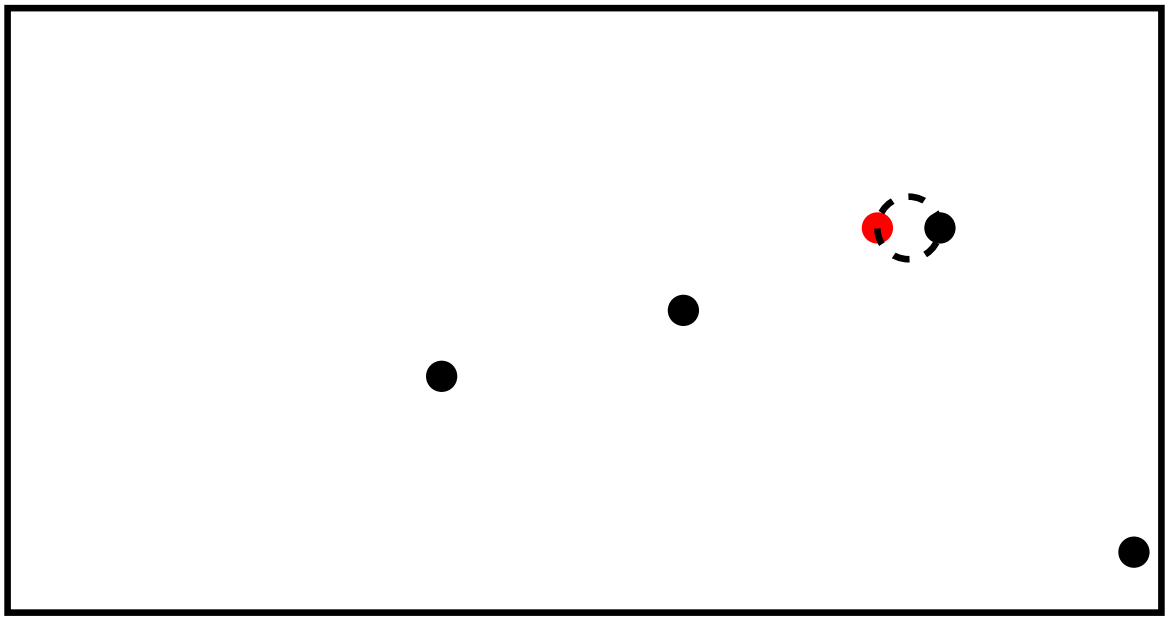

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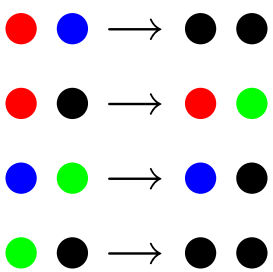

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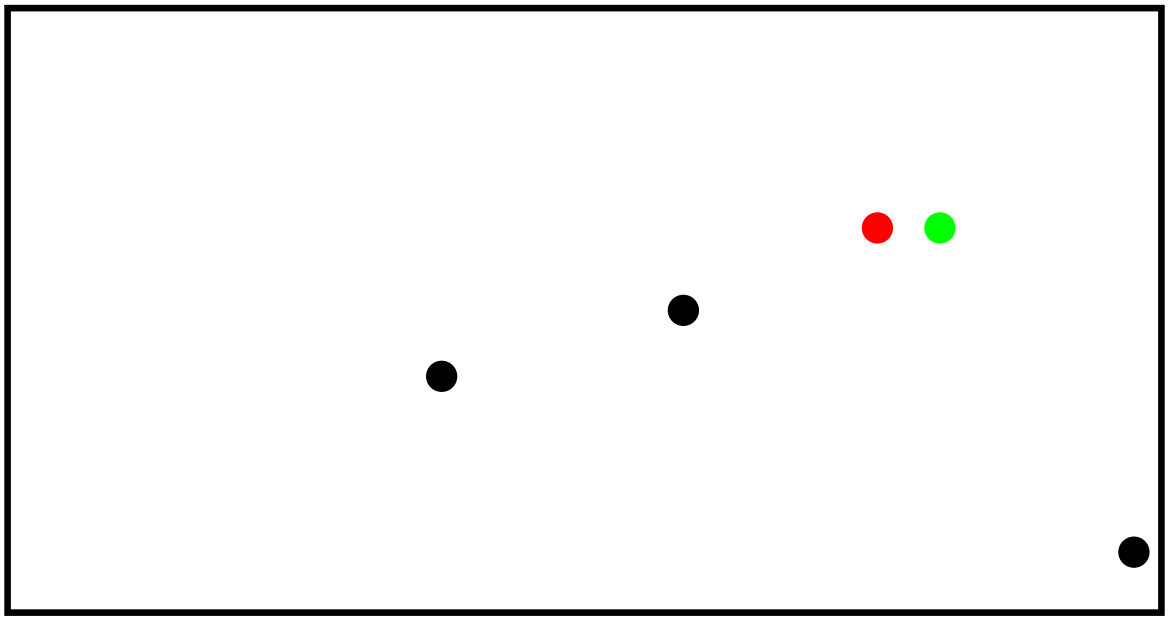

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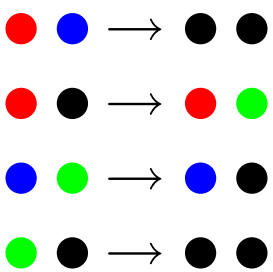

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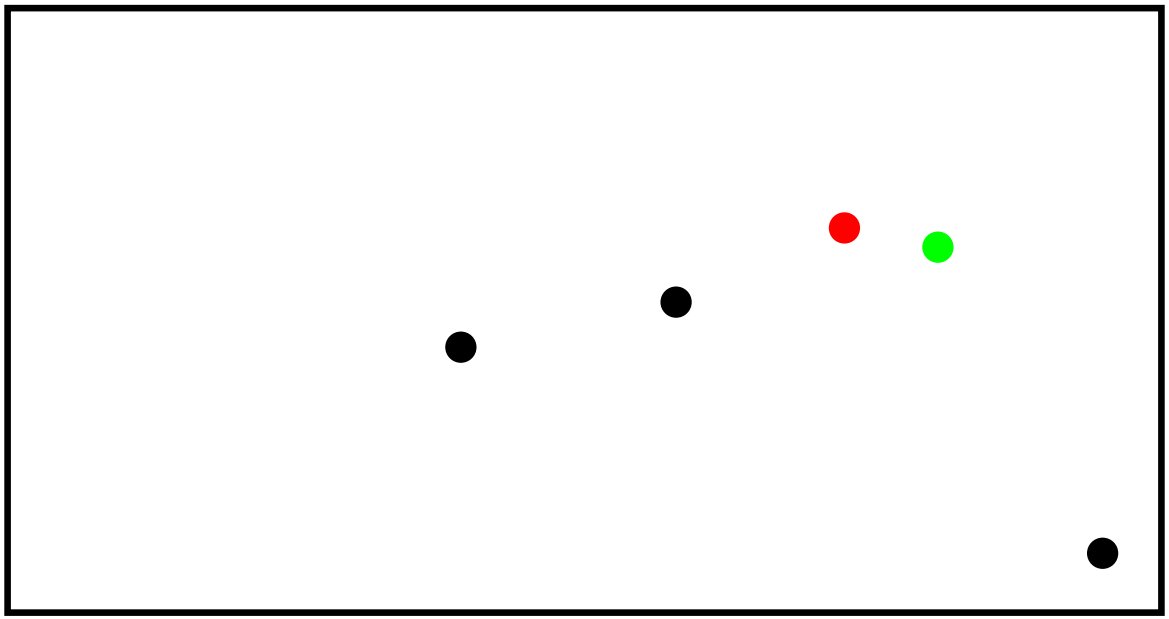

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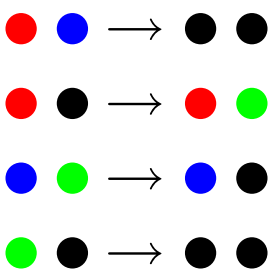

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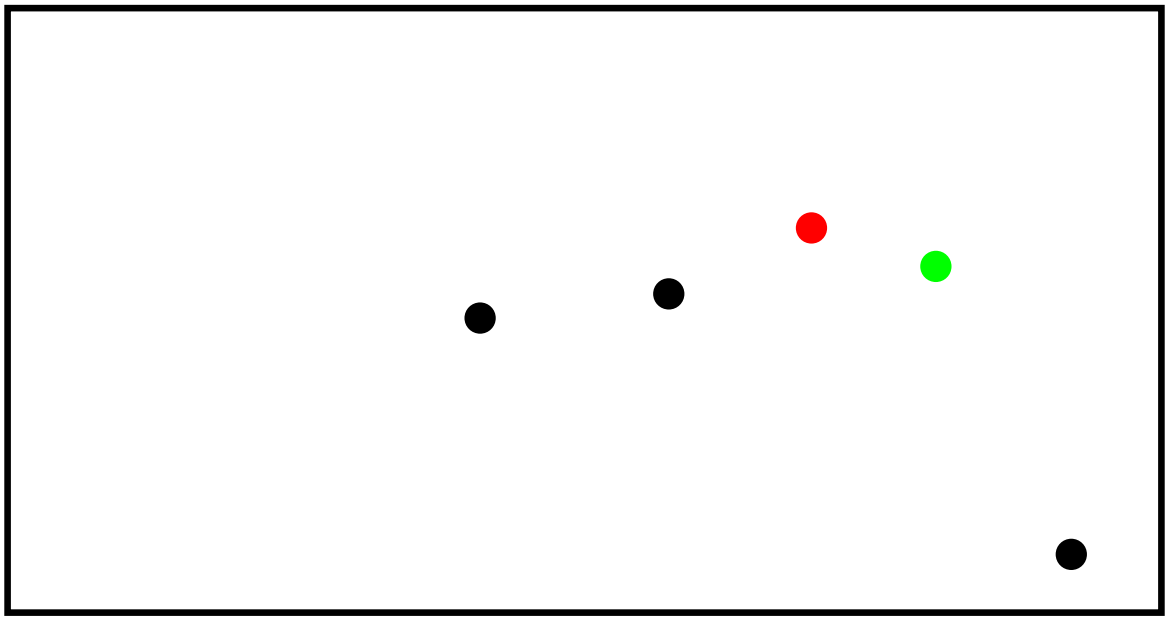

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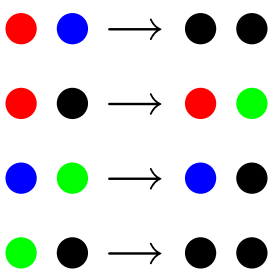
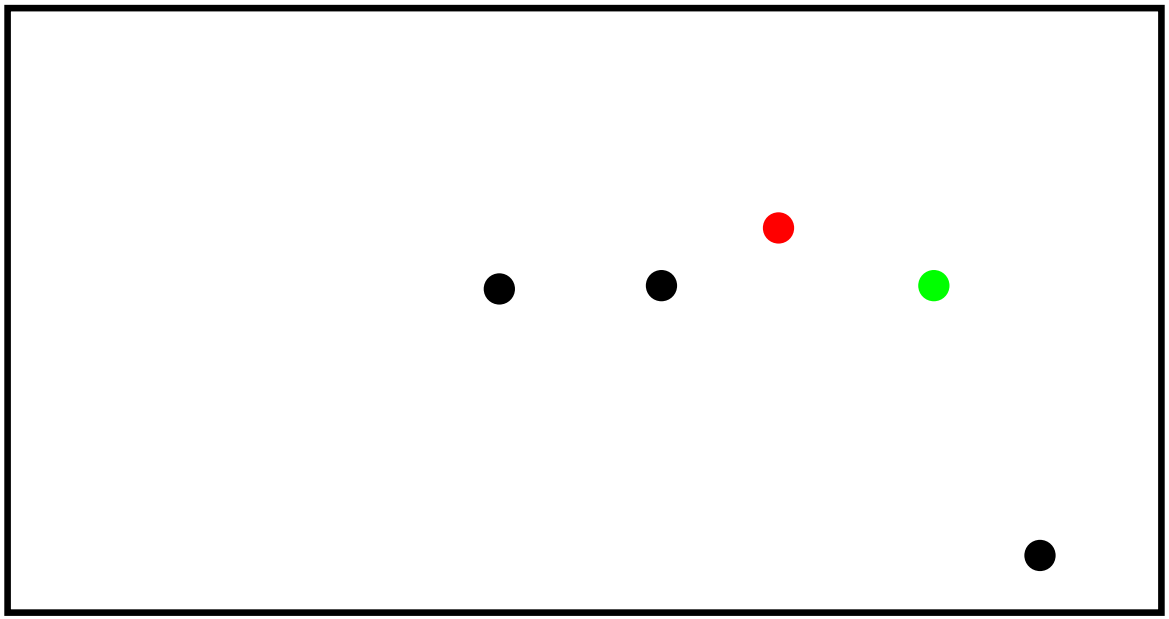


Example 2: starting with 2 blues, 3red



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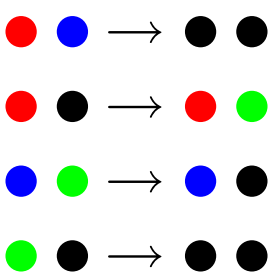
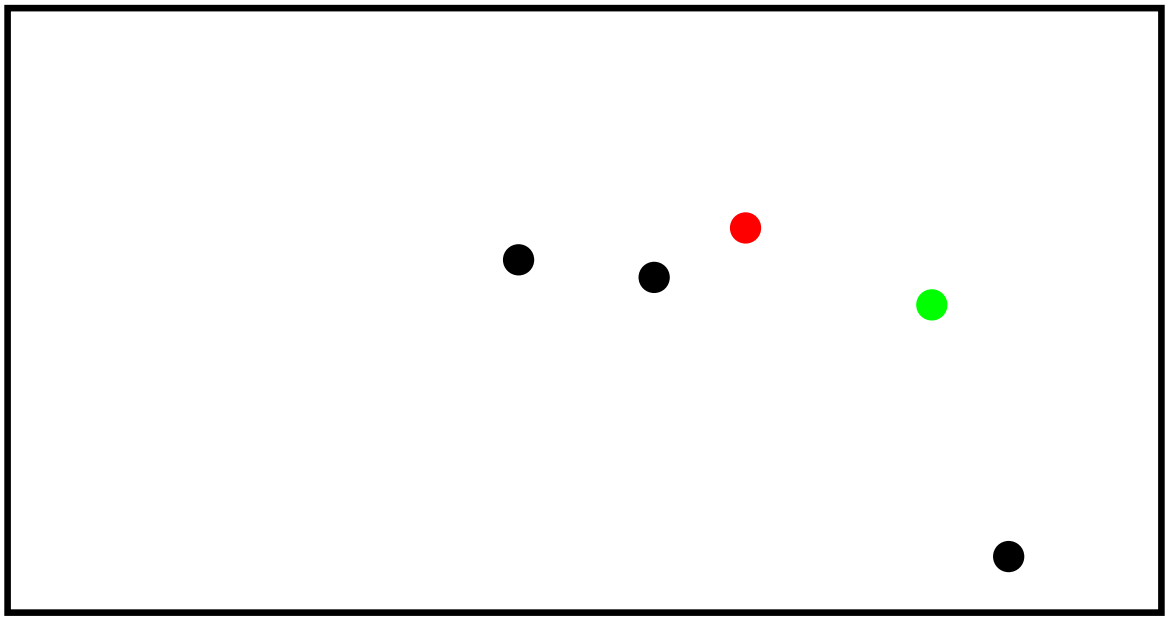


Example 2: starting with 2 blues, 3red



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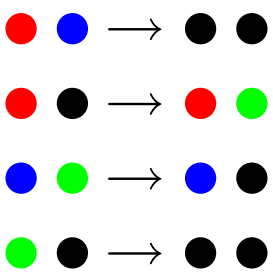

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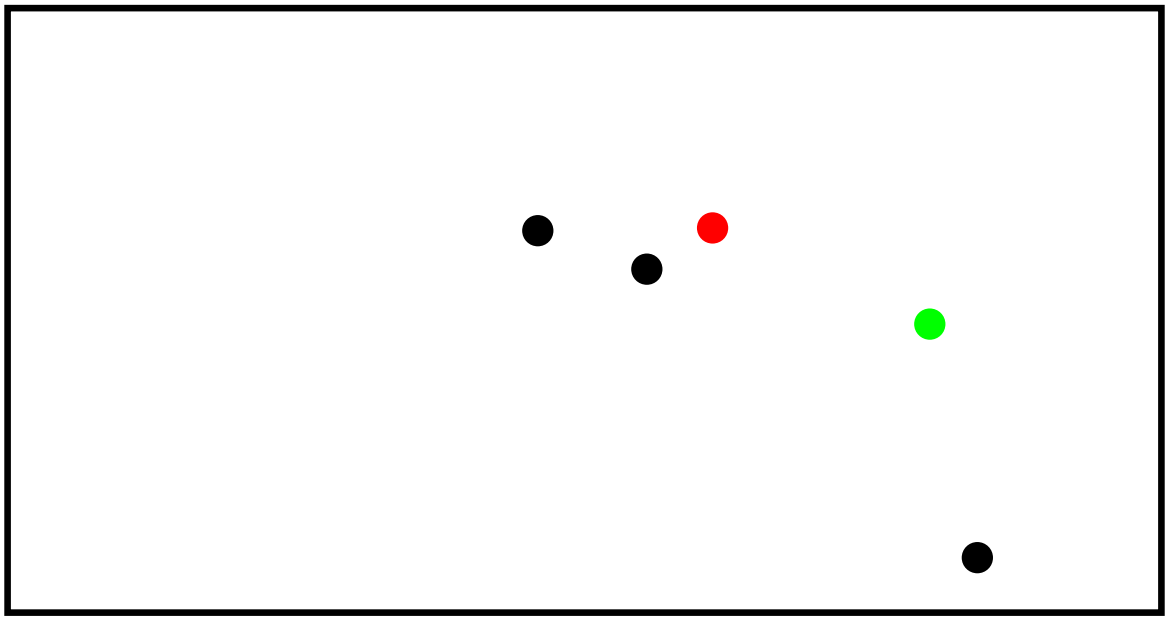

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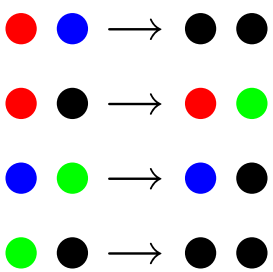

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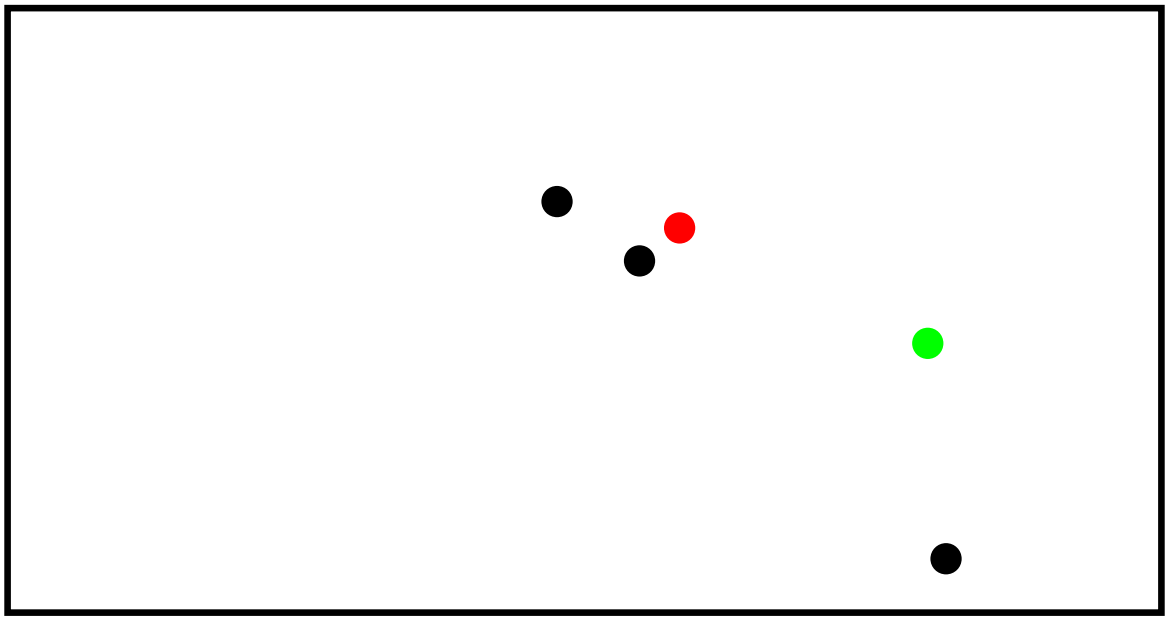

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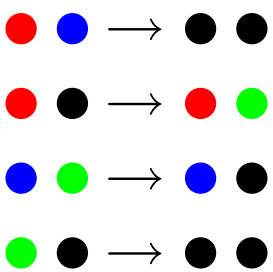

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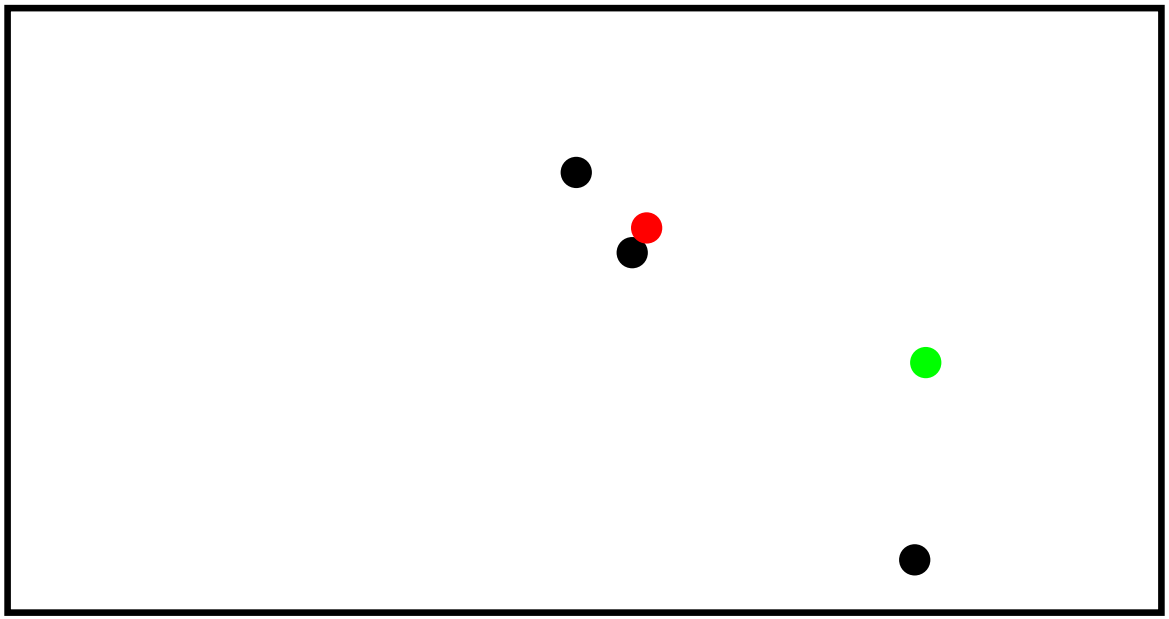

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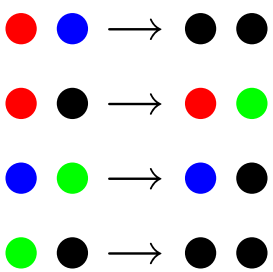

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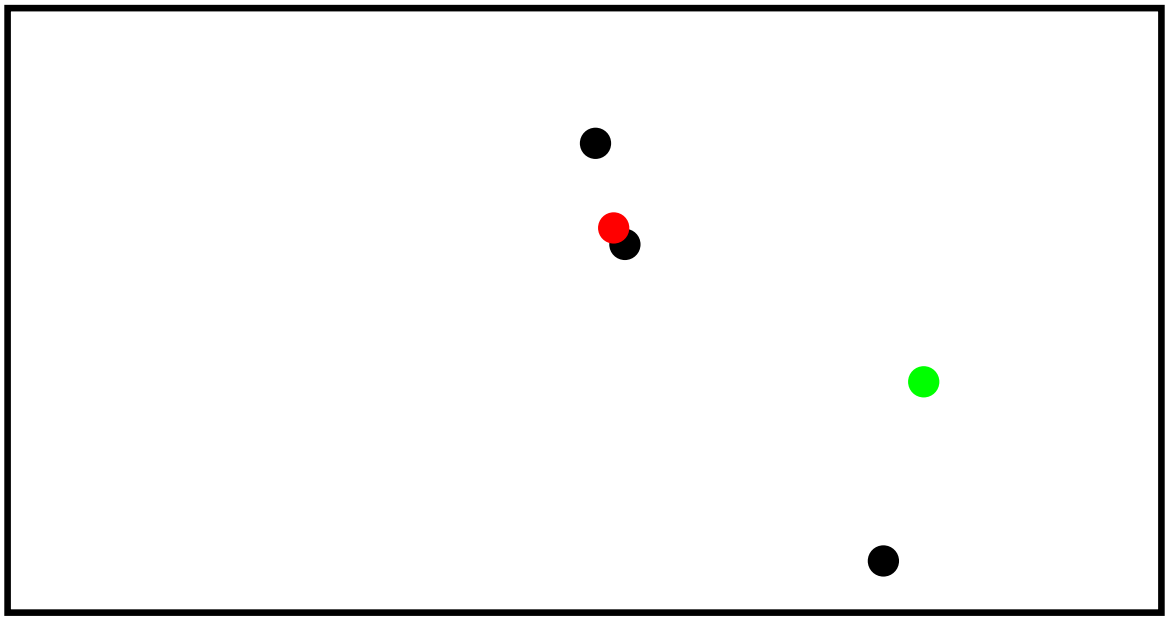

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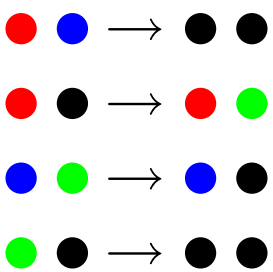

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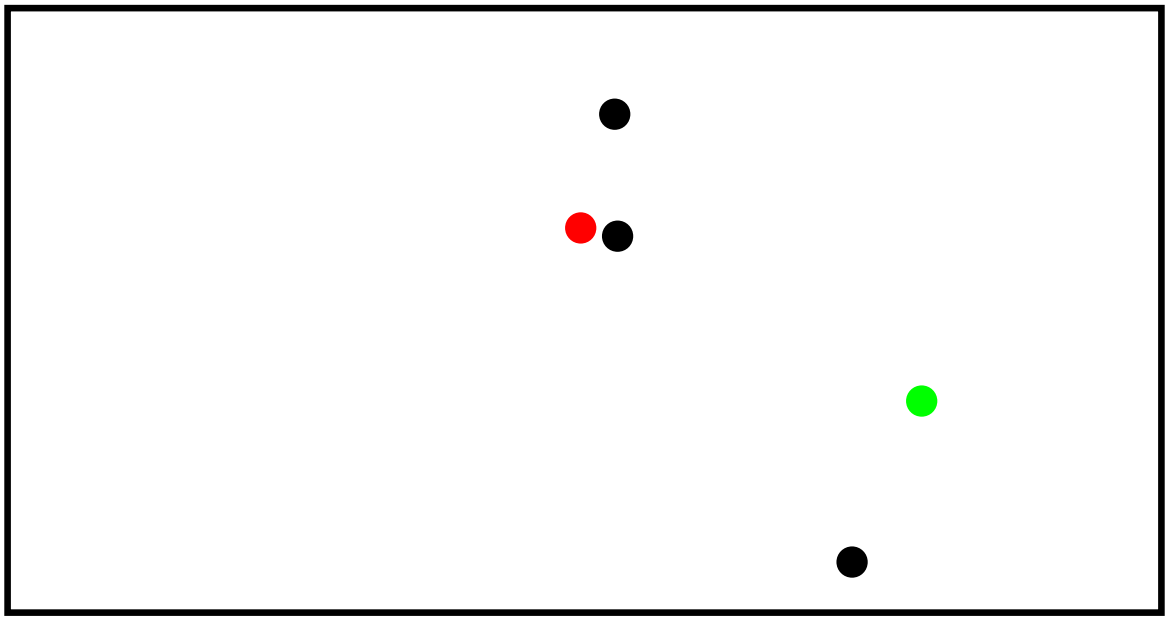

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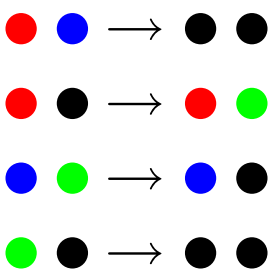

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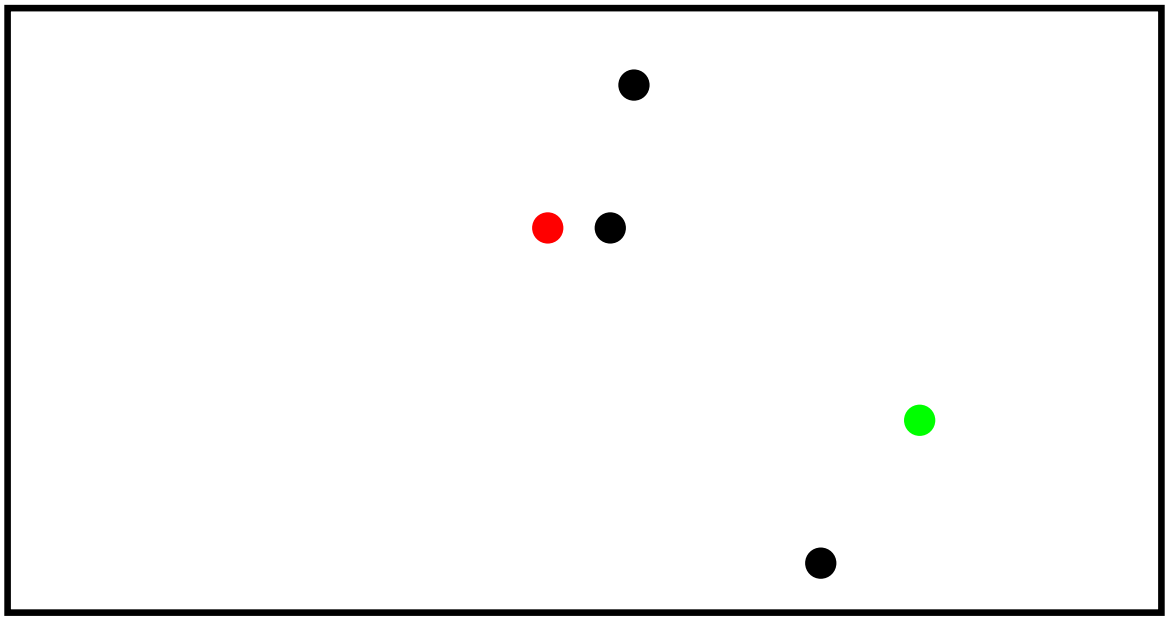

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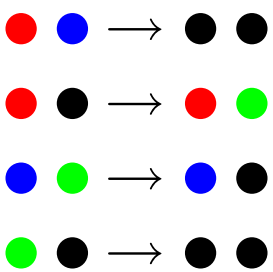

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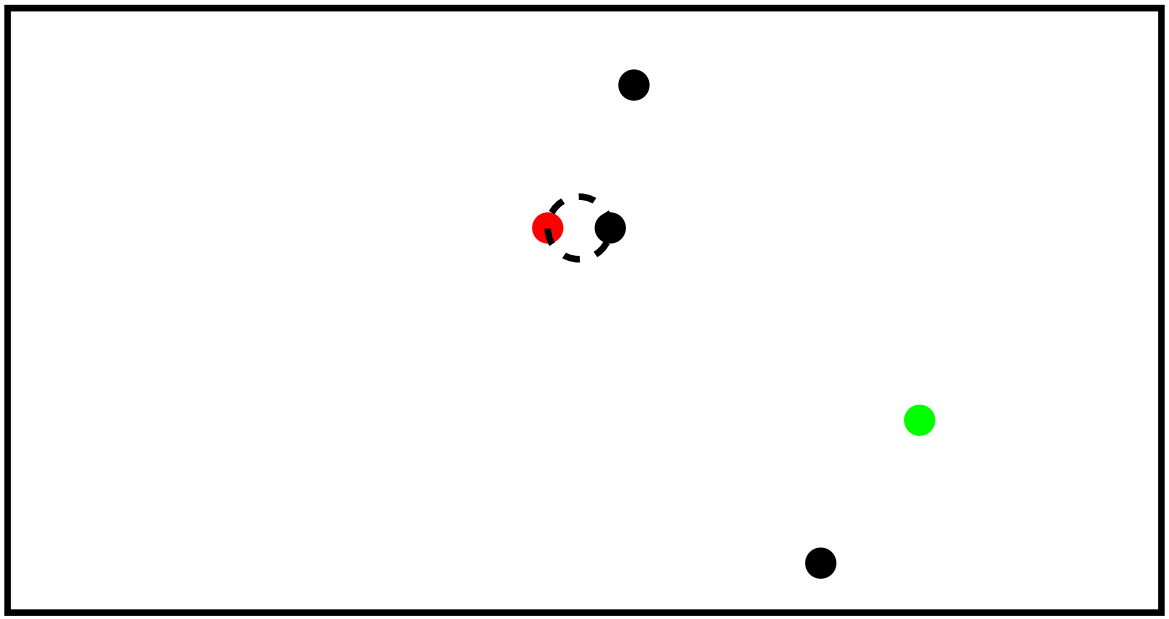

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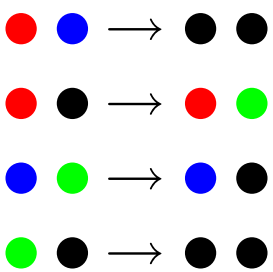

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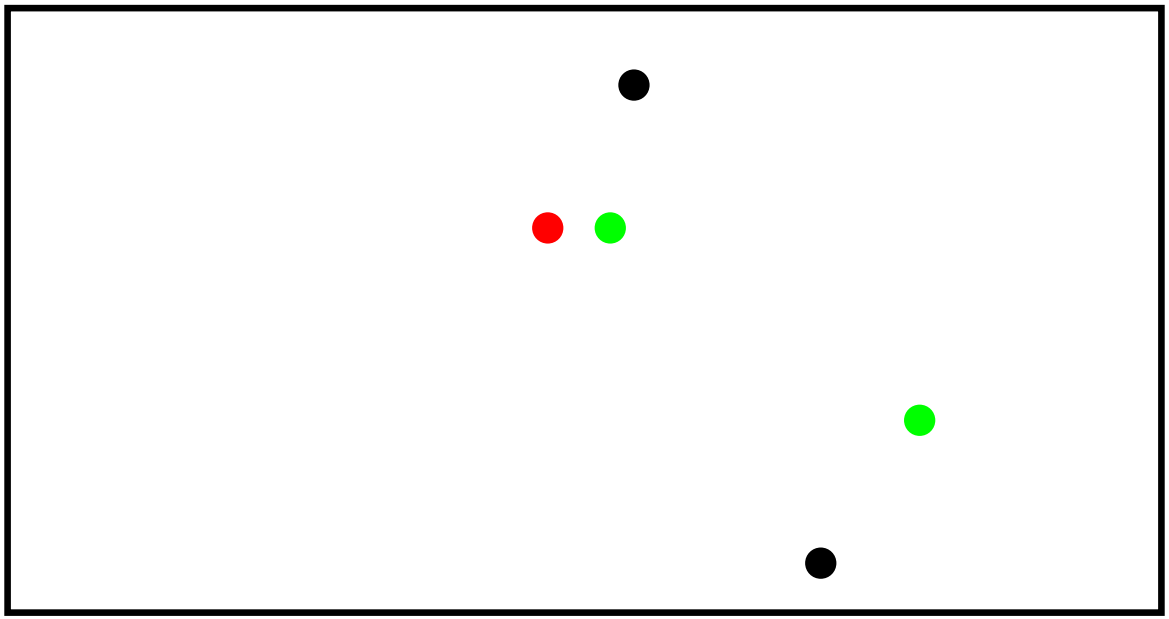

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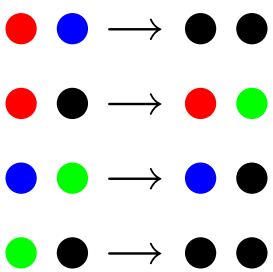

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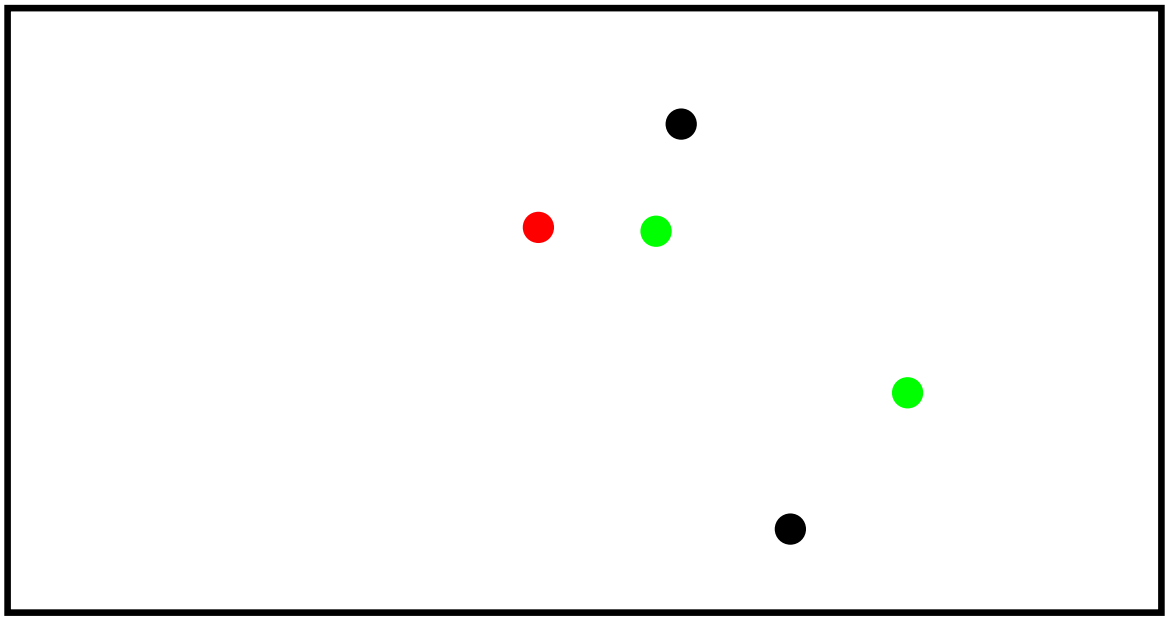

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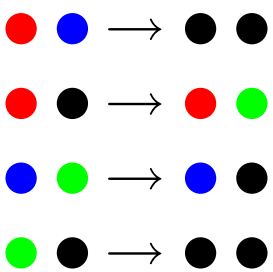

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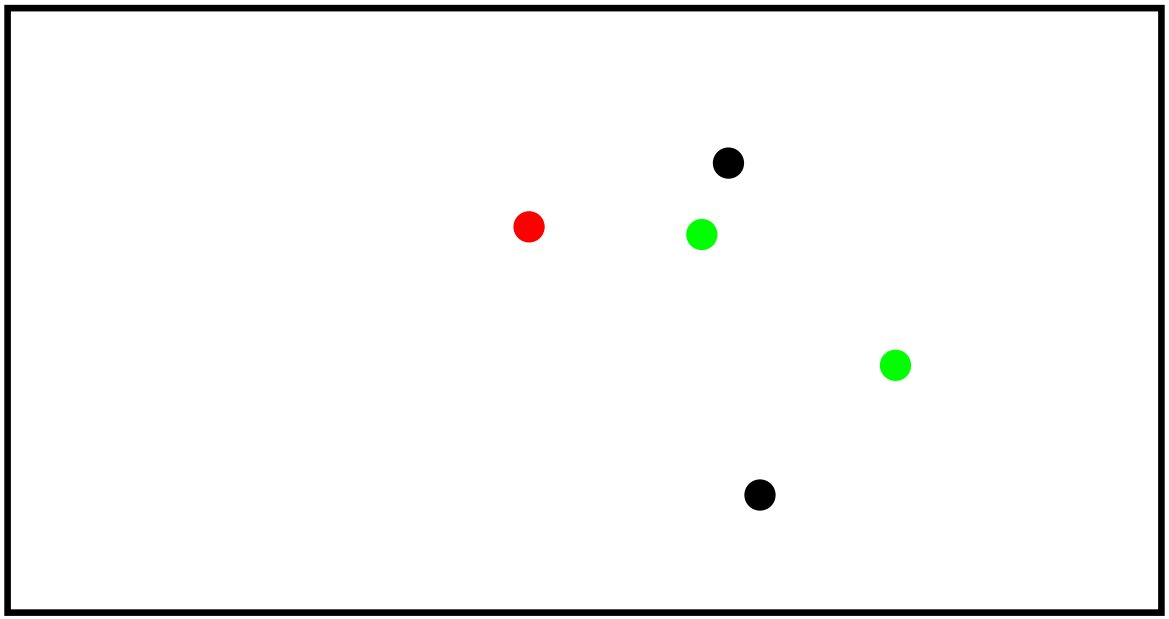

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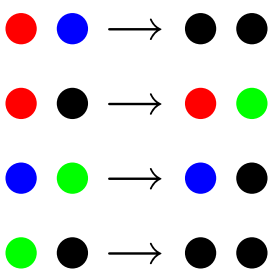

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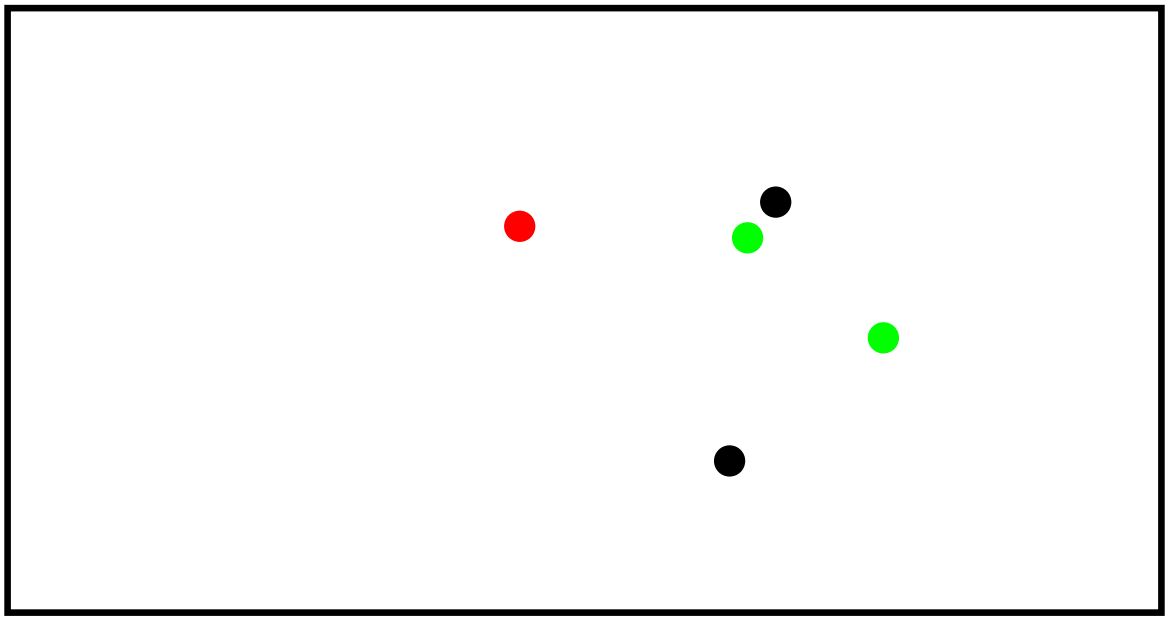

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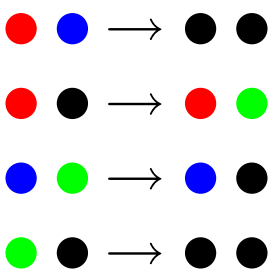

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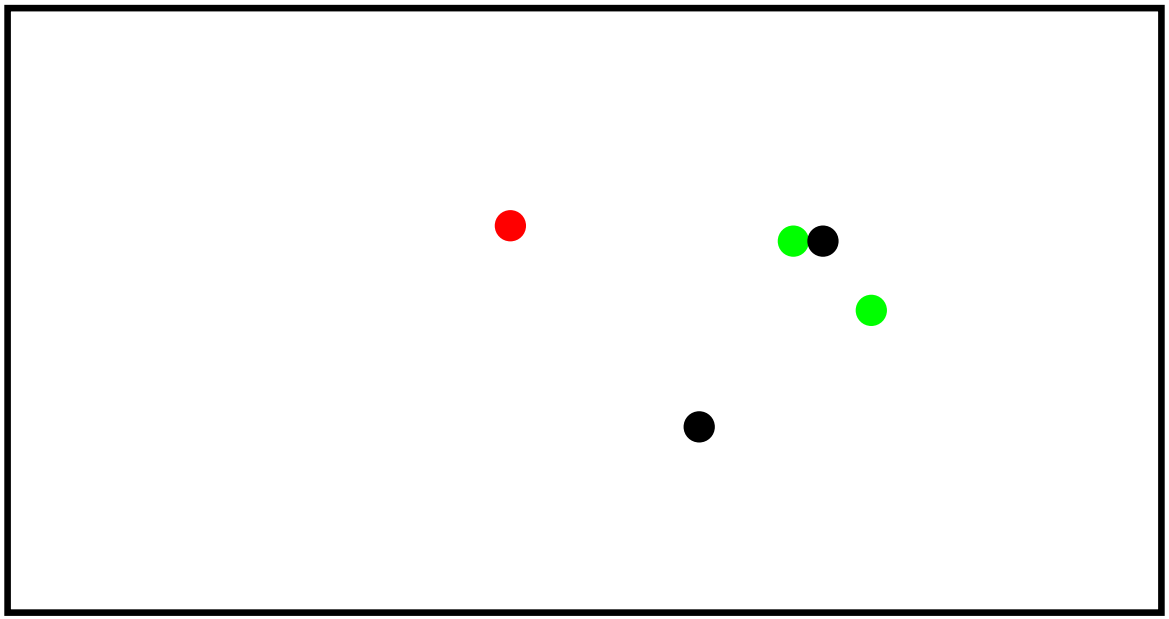

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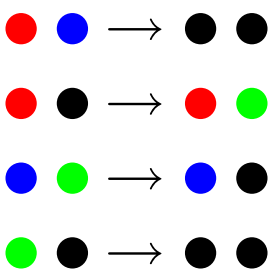

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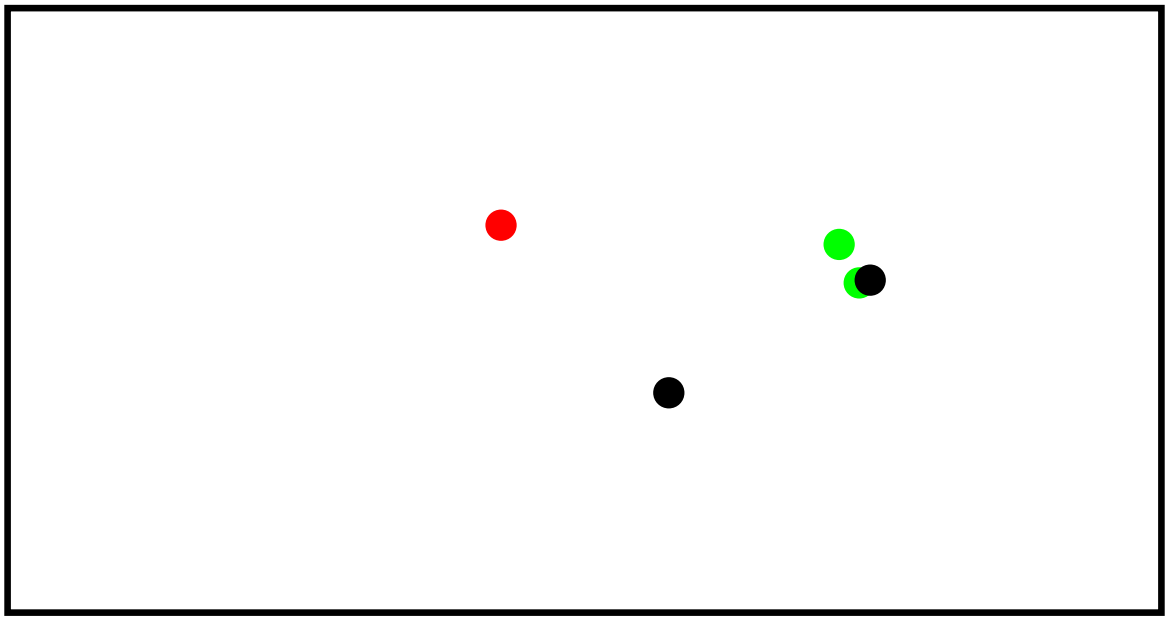

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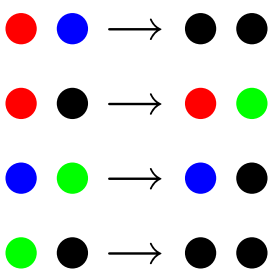

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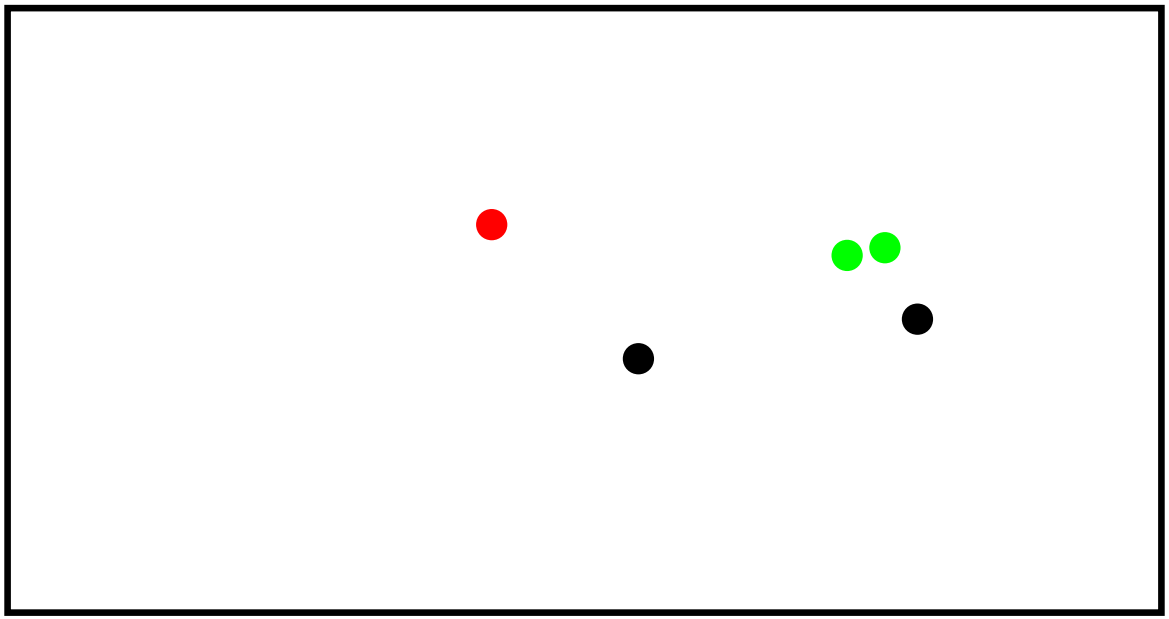

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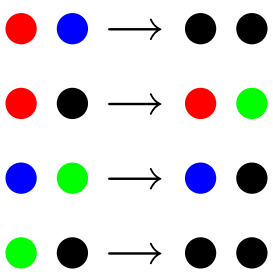

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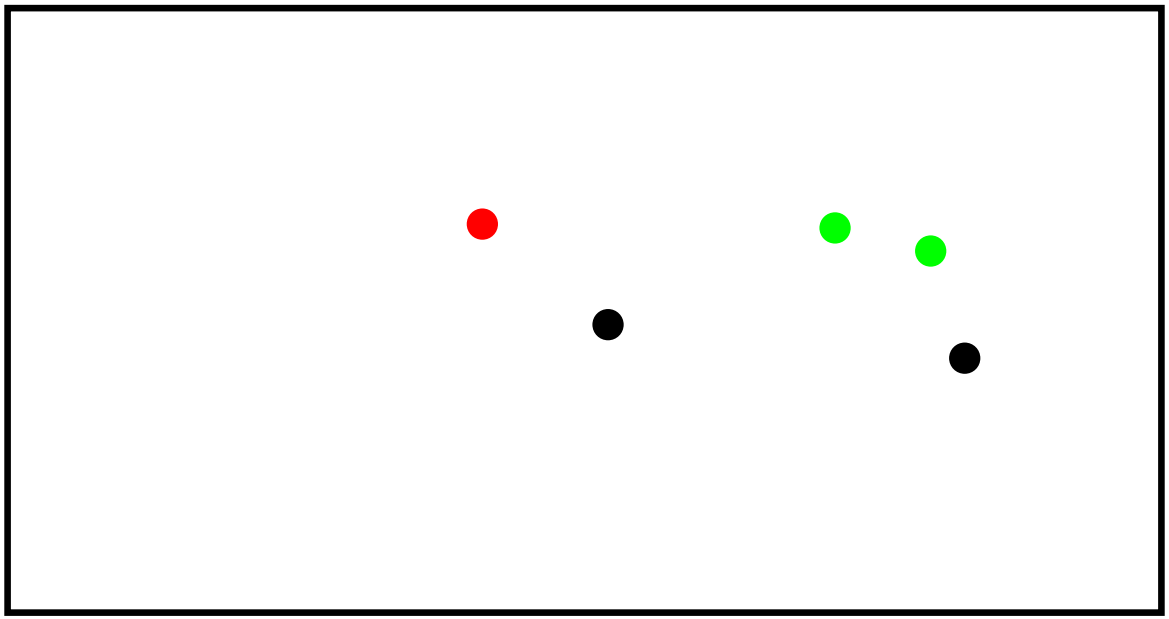

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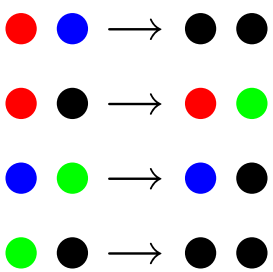

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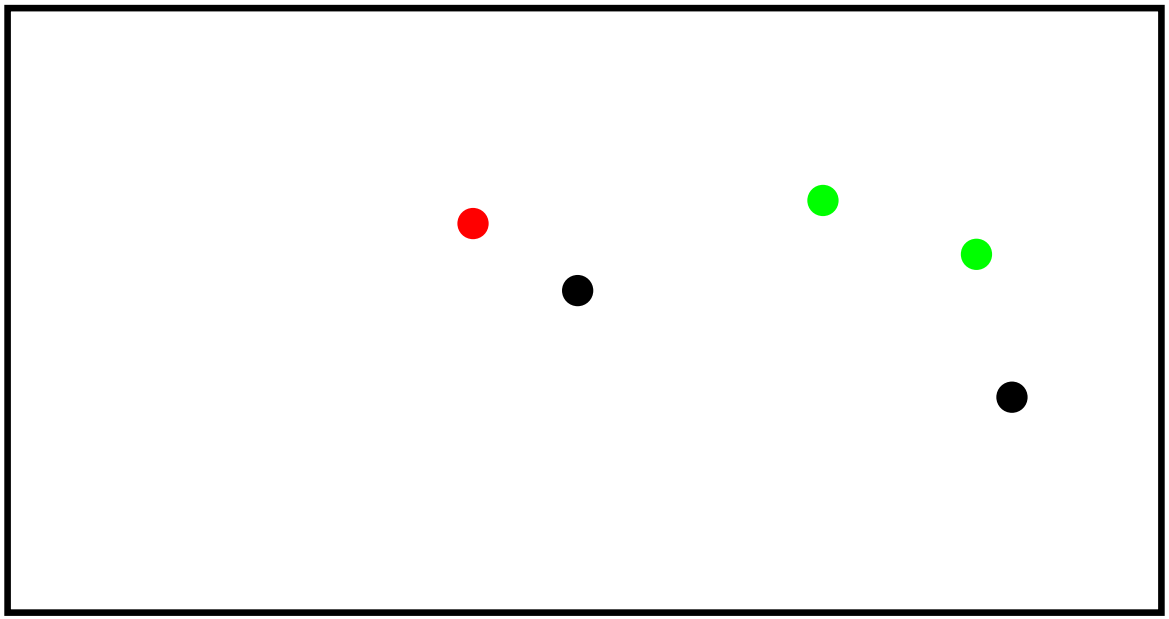

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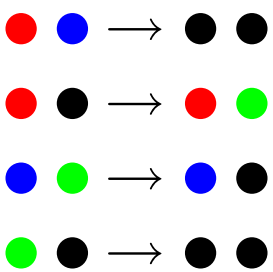

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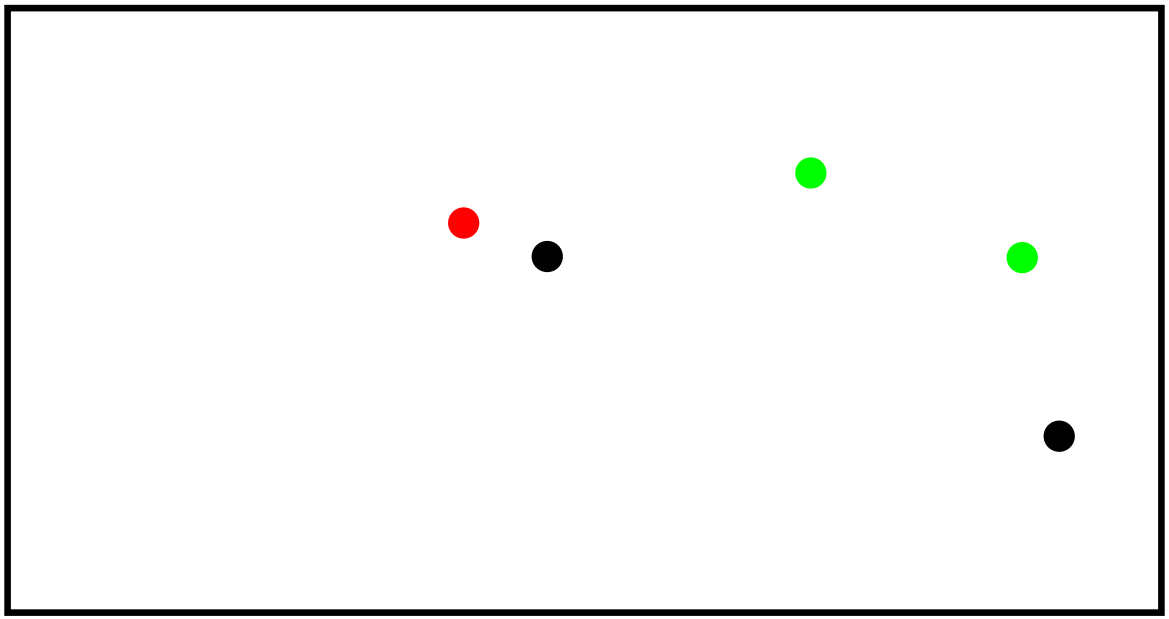

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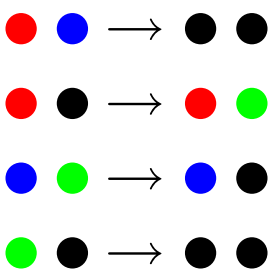

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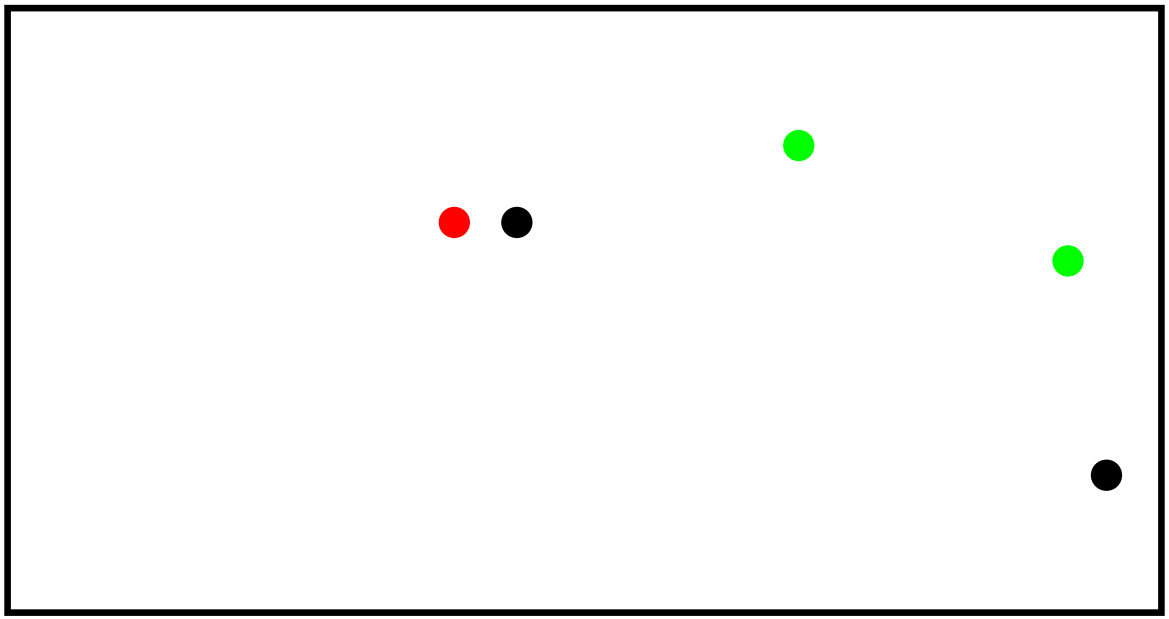

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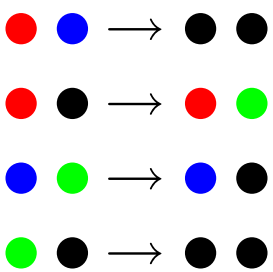

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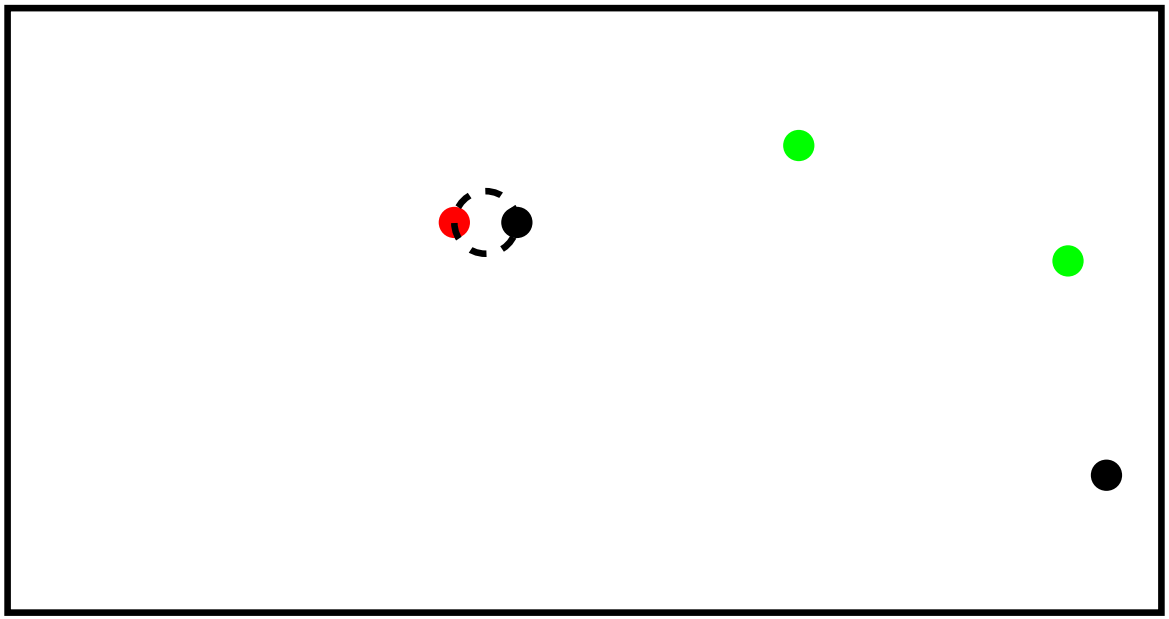

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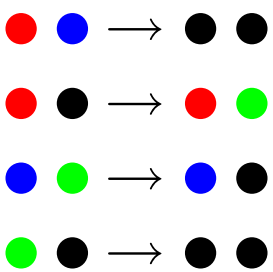

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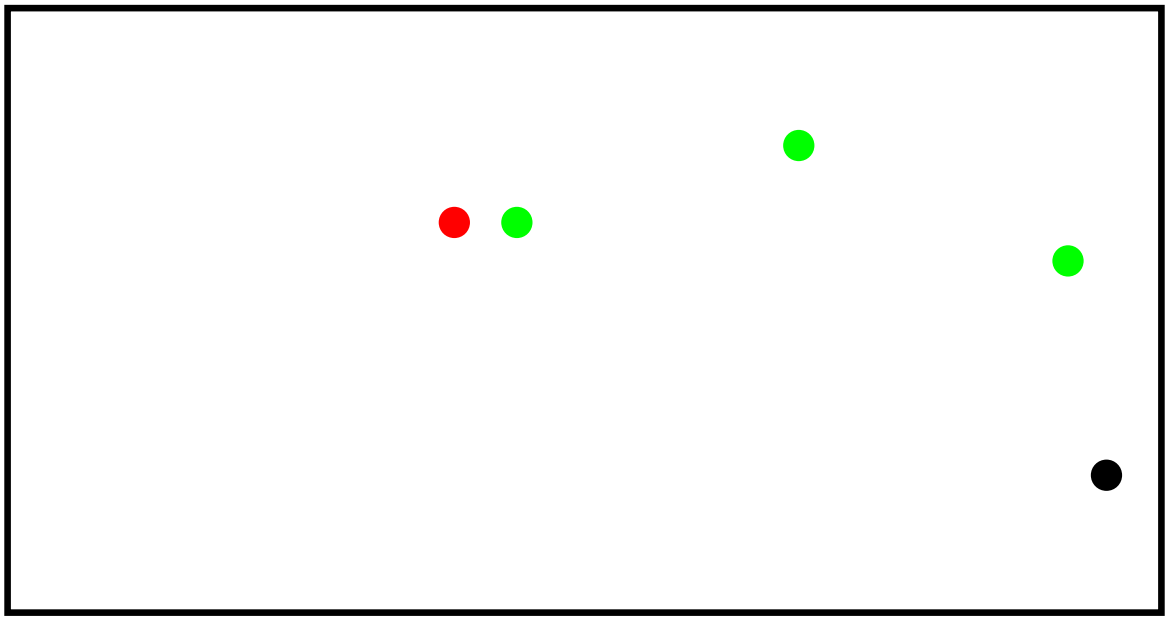

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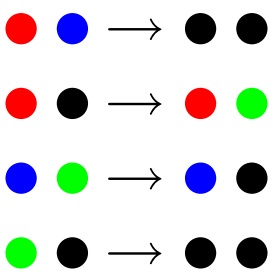

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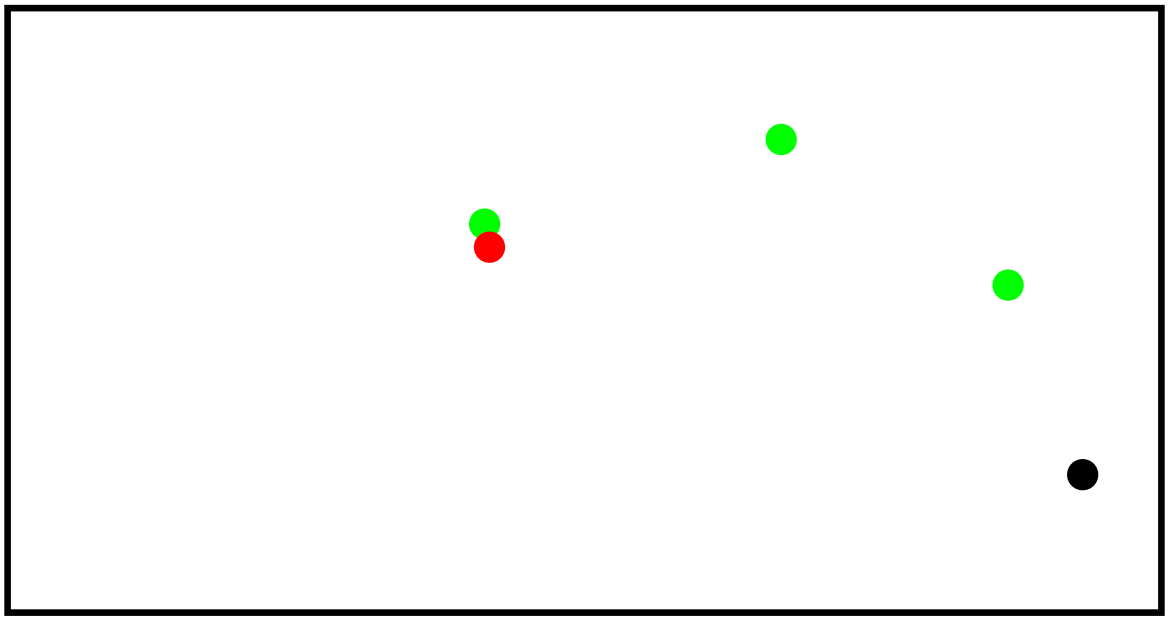

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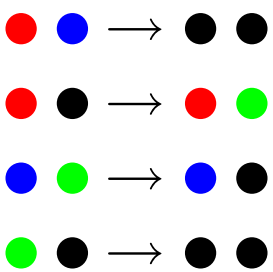

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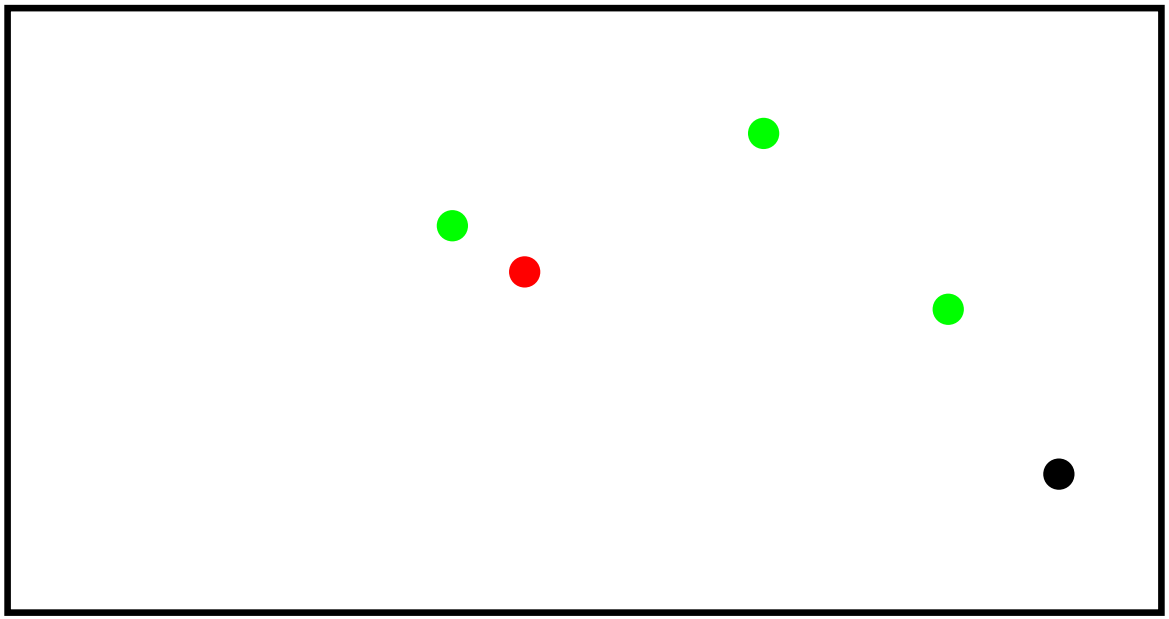

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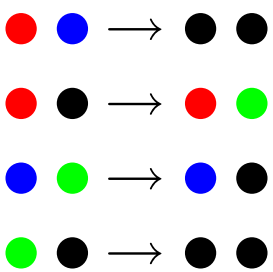

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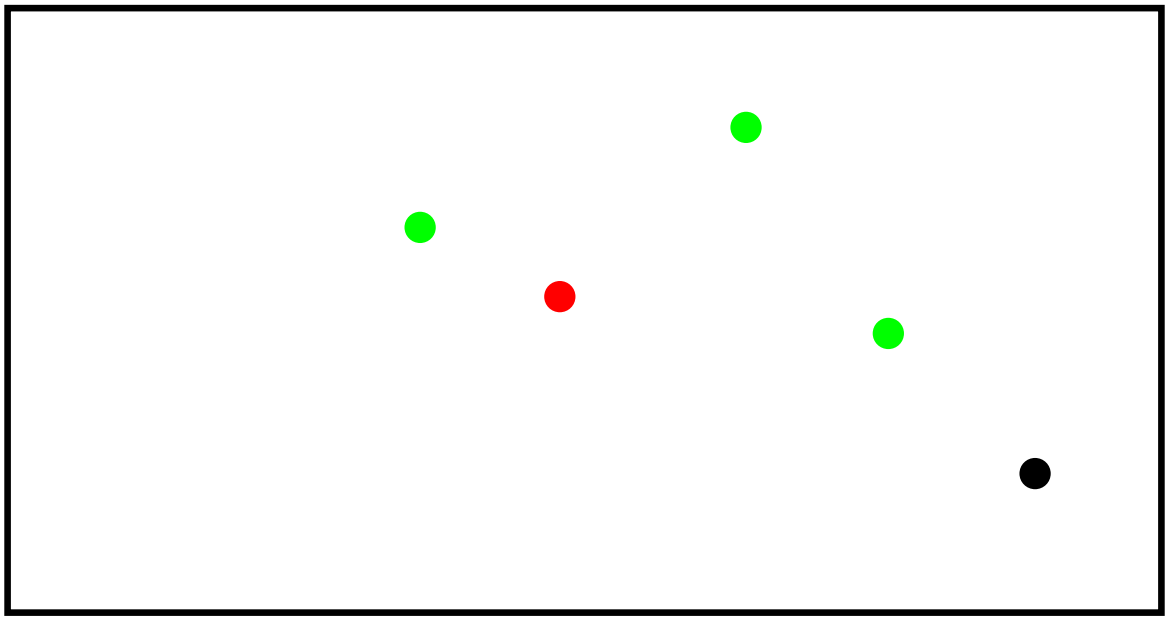

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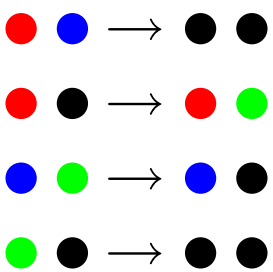

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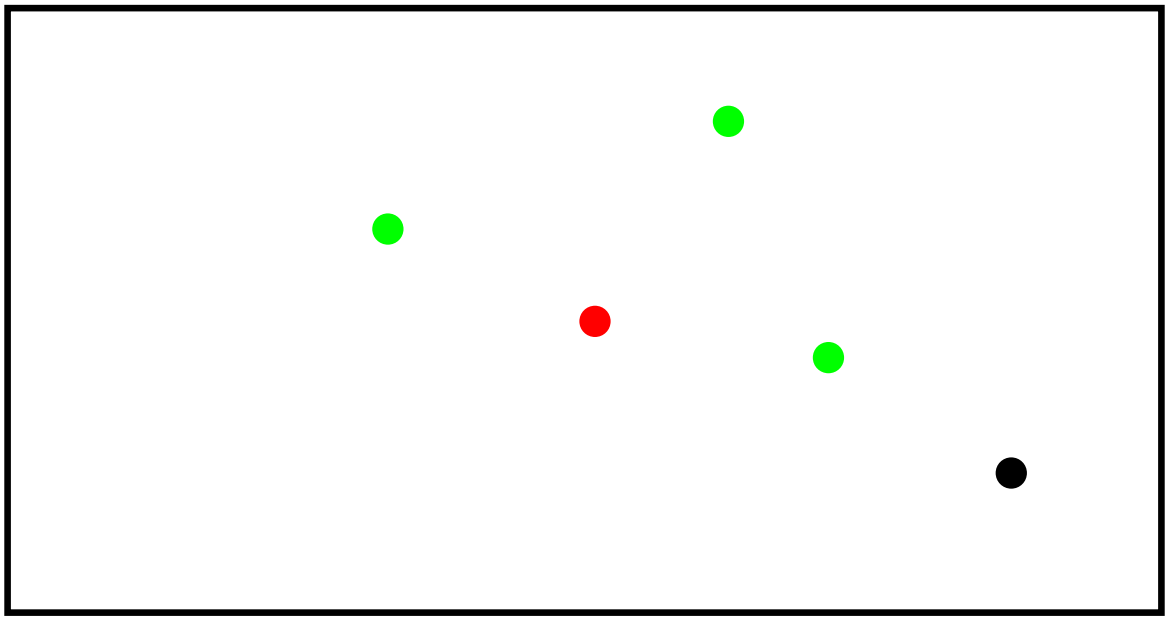

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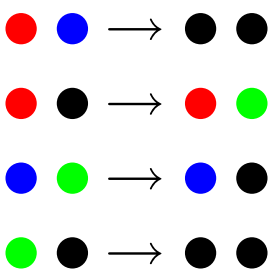

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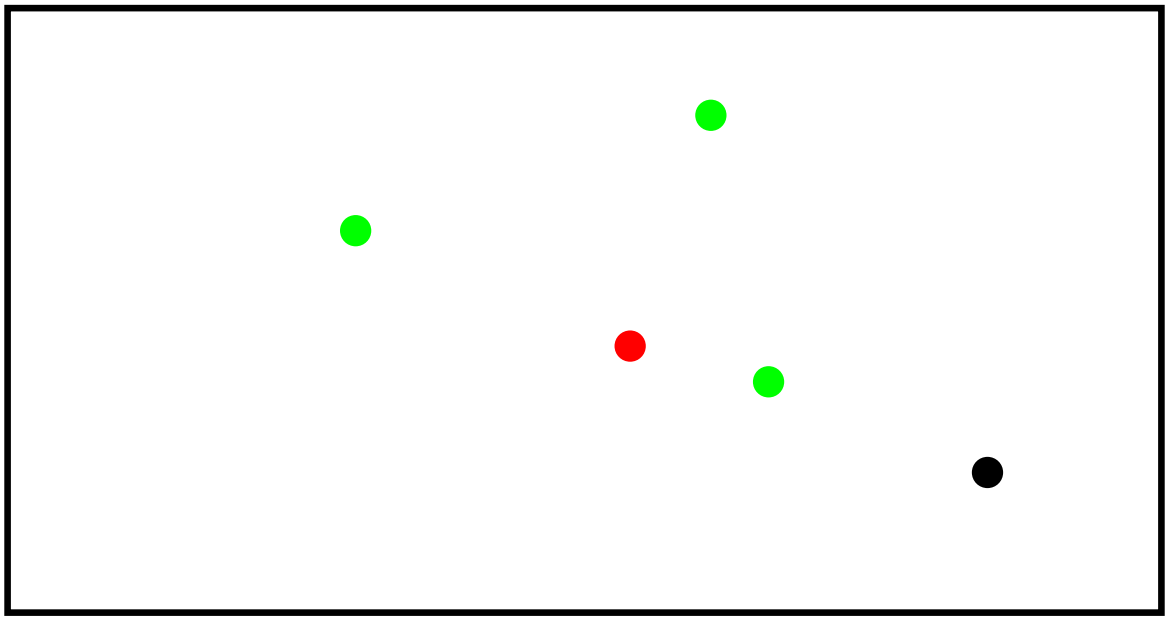

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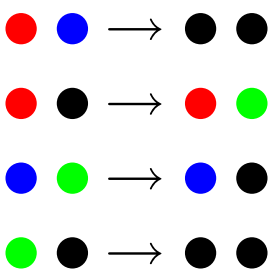

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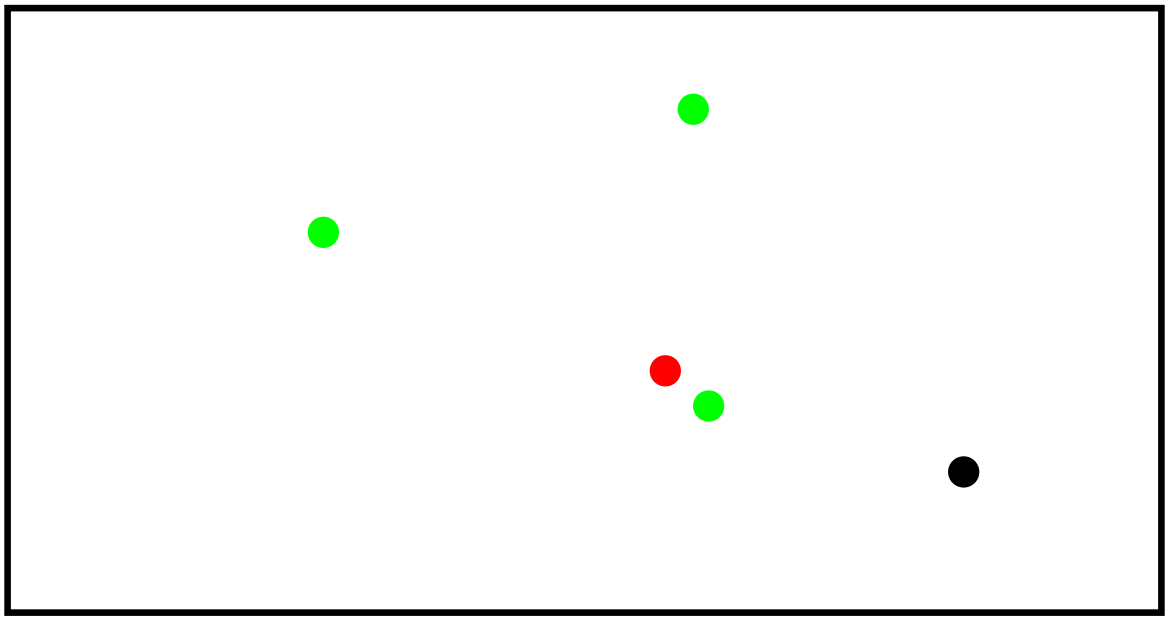

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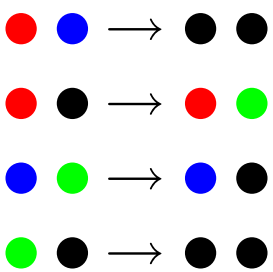

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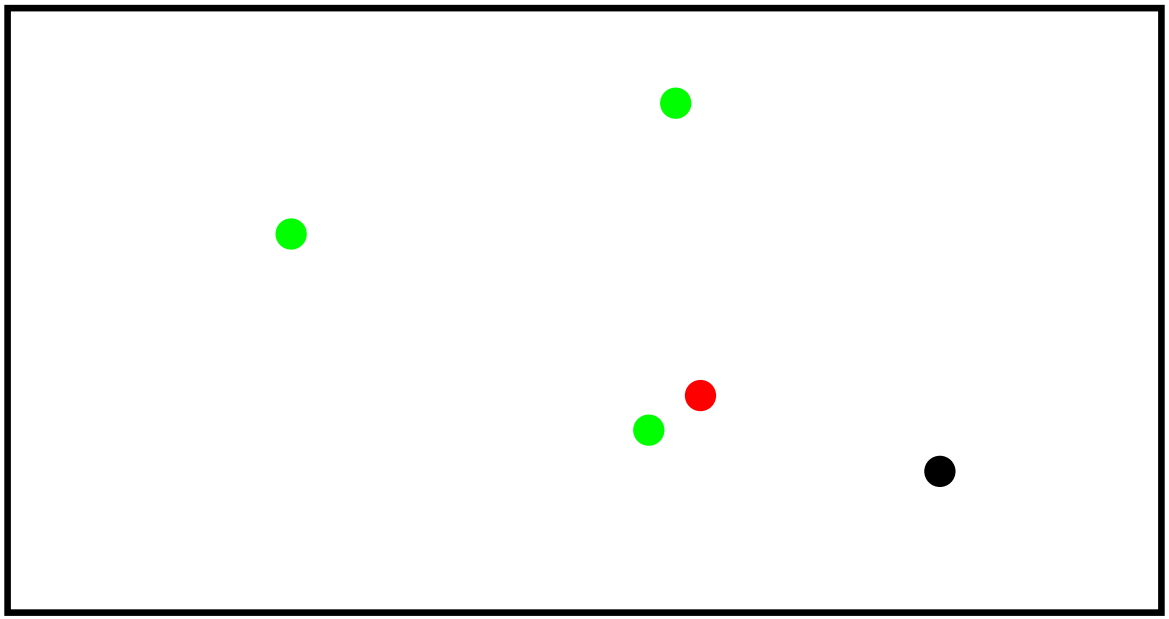
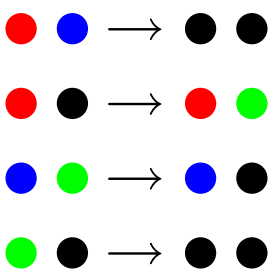
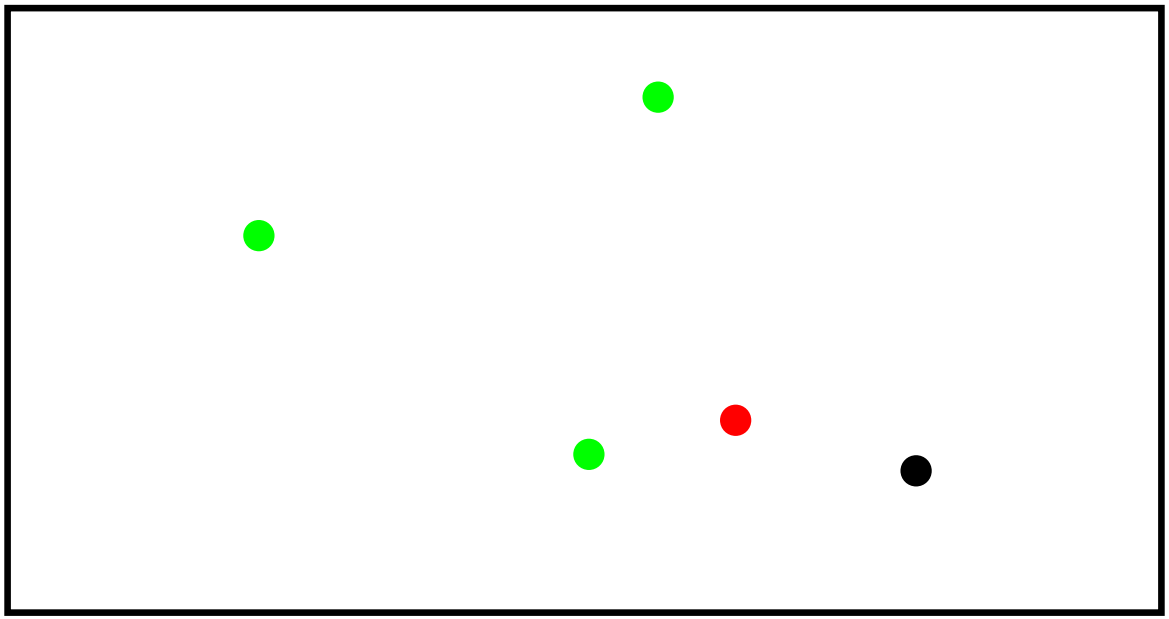


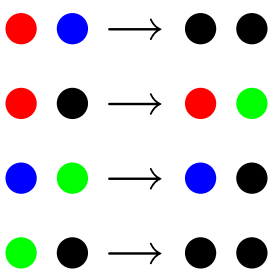
Program:



Example 2: starting with 2 blues, 3red



Program:



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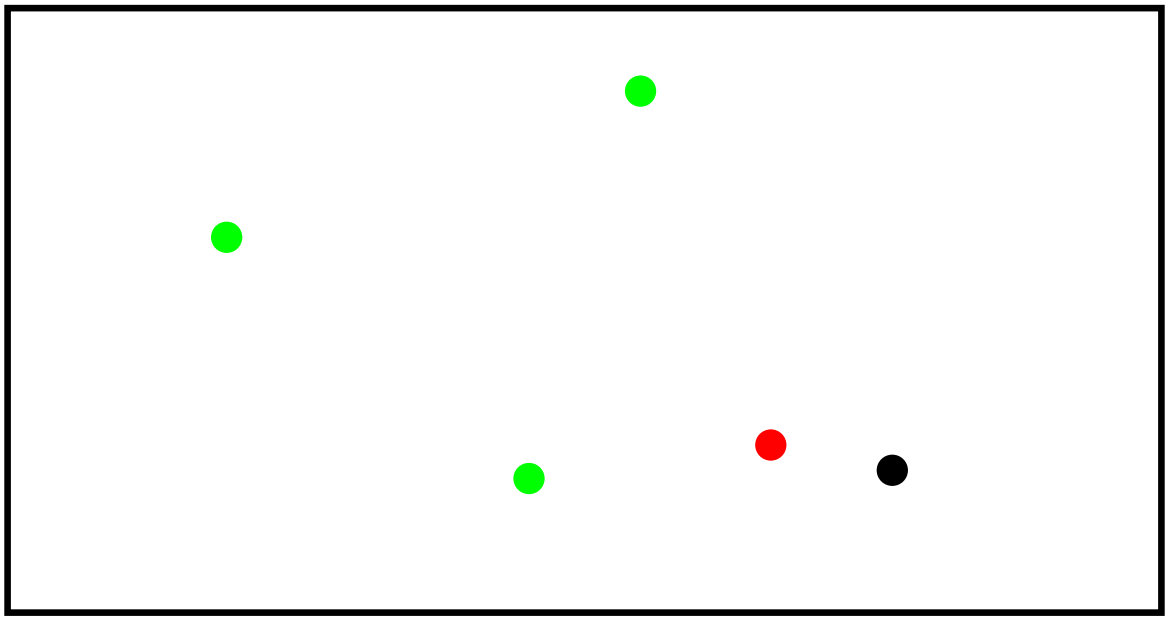

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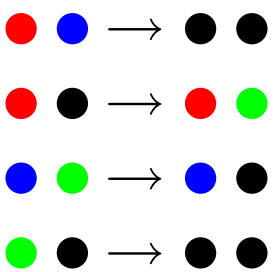

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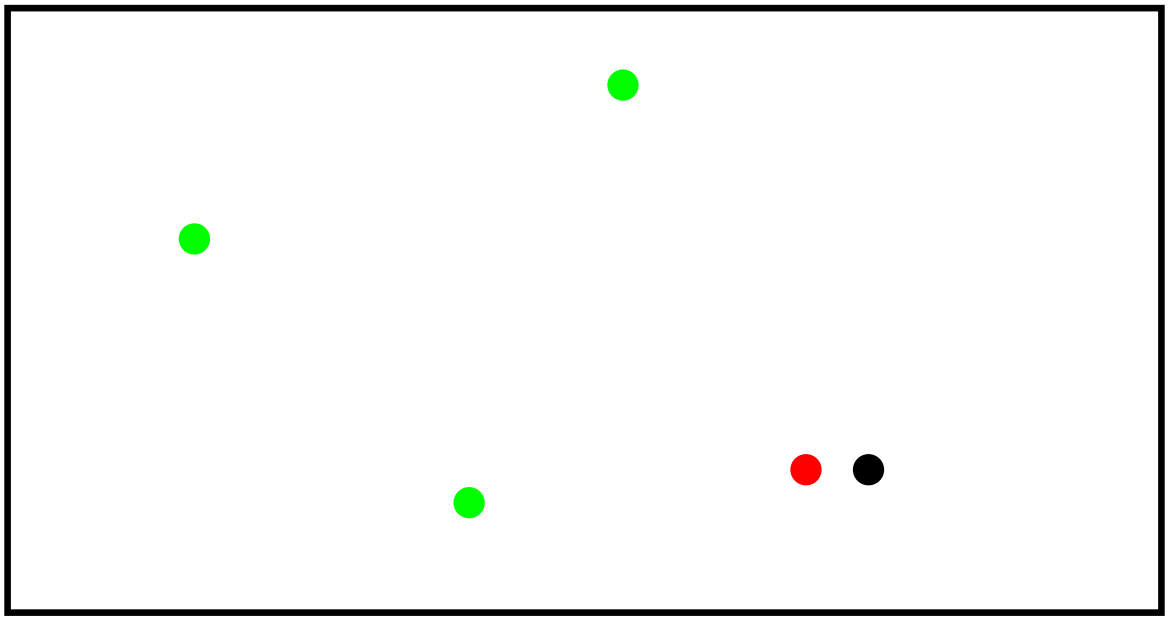
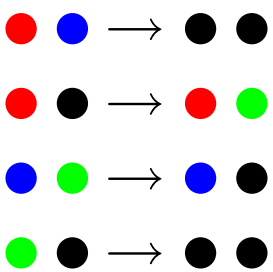


Program:



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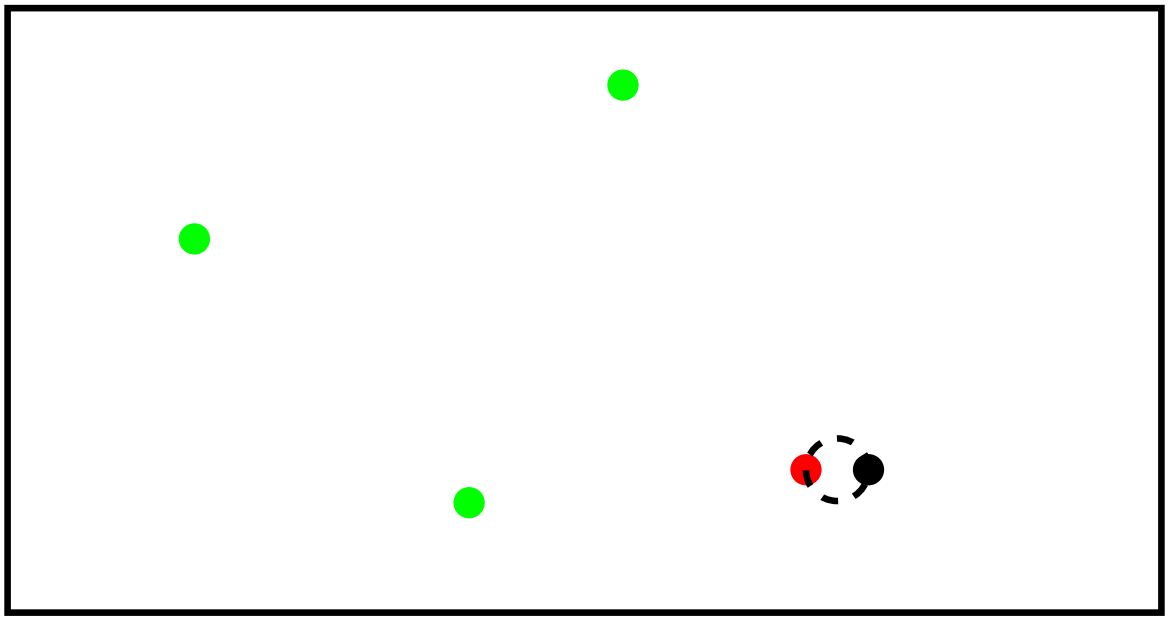

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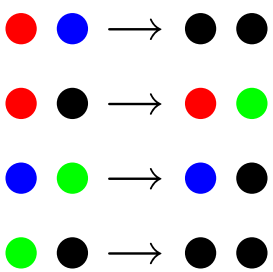

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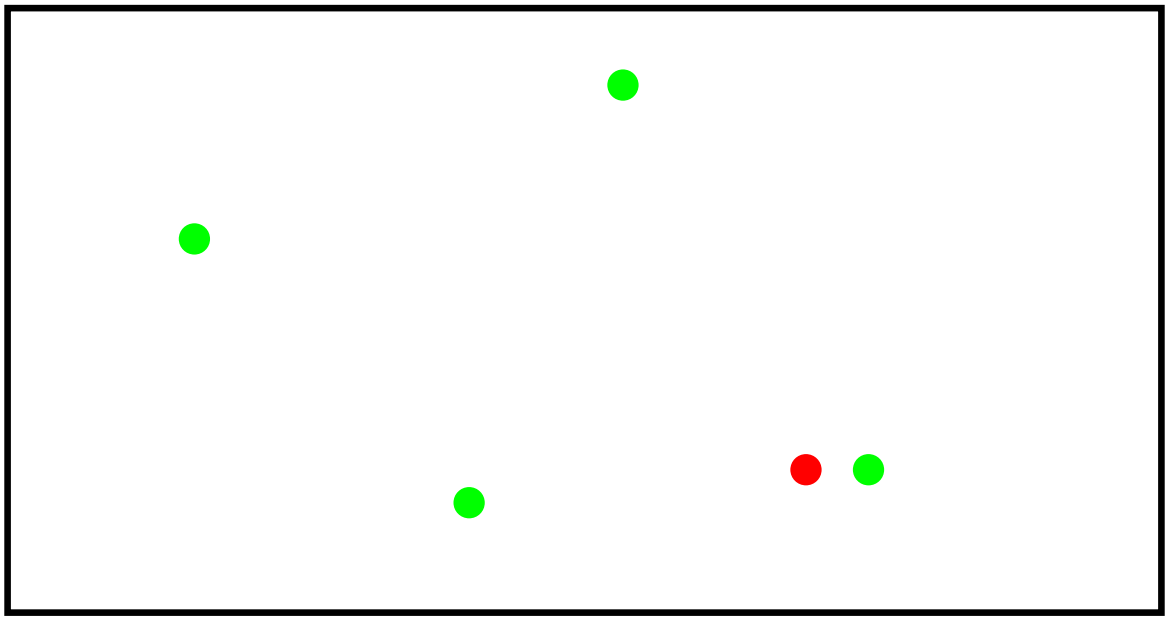
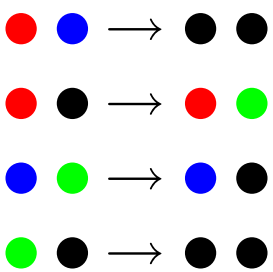
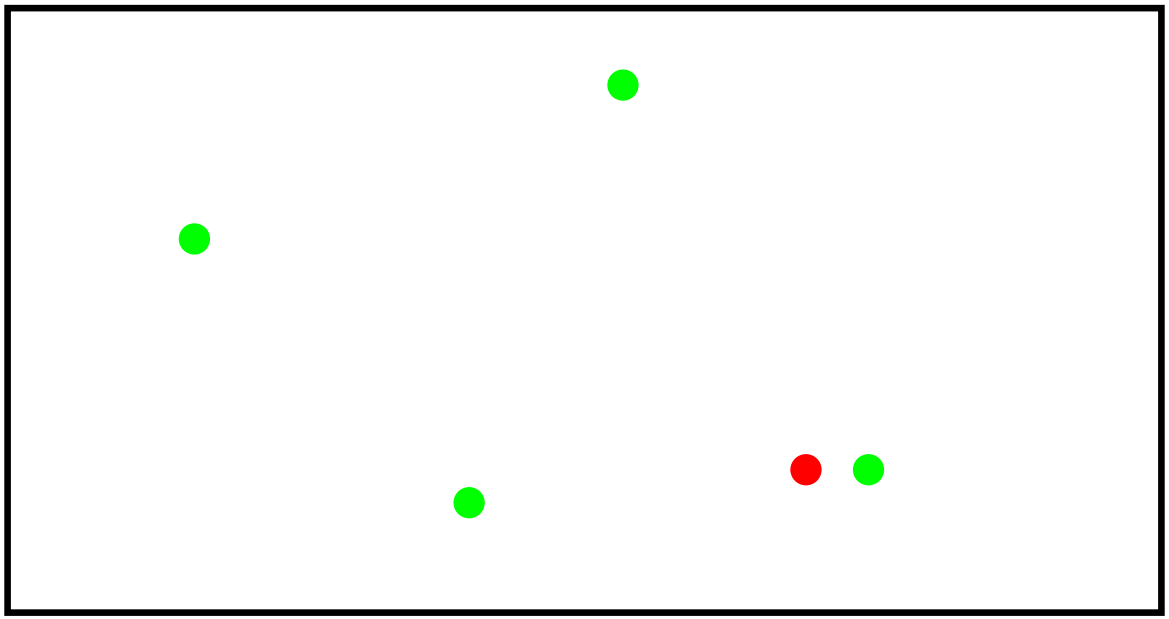


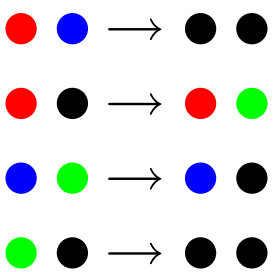
Program:



Example 2: Final result



Program:



What is computed?

Let's interpret

- ● and ● by *yes*.
- ● and ● by *no*.
- Whatever the initial state is, ultimately, all agents will agree.
- They agree on *yes* iff the initial population of ● is strictly greater than the initial population of ●.

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$$n_{●} > n_{●},$$

that is to say

$$n_{●} - n_{●} > 0.$$

Classical Population Protocols: Computable Predicates

Theorem (Angluin et al. 2006)

A predicate is computable iff it is on the following list:

- $\sum_{i=1}^{k} c_i n_i \geq a$, *where* a, c_i*'s are integer constants*
- $\sum_{i=1}^{k} c_i n_i \equiv a \pmod{b}$ *where* a, b *and* c_i*'s are constants*
- *Boolean combinations of the above predicates*

Alternate Characterization: Presburger Arithmetic

A predicate is computable iff it can be expressed in first-order logic using the symbols $+, 0, 1, \vee, \wedge, \neg, \forall, \exists, =, <, (,)$ and variables.

(This system is known as Presburger Arithmetic [1929].)

Note: no multiplication.

Examples:

majority: $x_0 < x_1$

divisible by 3: $\exists y : y + y + y = x_1$

at least 40%:$x0 + x0 < x1 + x1 + x1$

Menu

Inspiration: Classical Population Protocols

Large Population Protocols

Discussion and Conclusion

Population protocols with large populations

We want to consider

- a large population of agents (instead of finitely many agents)
- proportions of agents in a given state (instead of counts)
- protocols as computing functions (not only predicates)

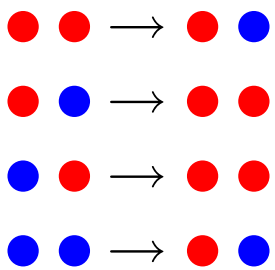
Sub-menu

A scenario

- Consider a population of individuals provided with mobile phones, moving in a somewhat restricted environment.
- Whenever two individuals meet, i.e., whenever they get within the transmission range of each-other, their respective mobile devices get connected, and potentially exchange data of common interest.
- In order to limit the amount of energy spent in such a process, one may want to limit the frequency of exchanges between nodes:
  - one could let two connected nodes exchange information only if they are both in some specific state, called *active*
  - say we want the frequency (or probability) of exchanges to be $p = \frac{1}{2}$

An example

States: {●,●}
Rules of interactions:

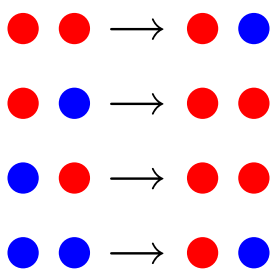


What can we say about

$$p_{\bullet} = \frac{number\ of\ \bullet}{number\ of\ \bullet + number\ of\ \bullet}?$$

Exercice 0: when the number n of agents is fixed.

- The interaction rules describe an homogeneous Markov chain with 2^n states:
  - Special configuration: $(\bullet, \bullet, \cdots, \bullet)$ is left in one step.
  - Any other configuration is reachable with positive probability from any other configuration.

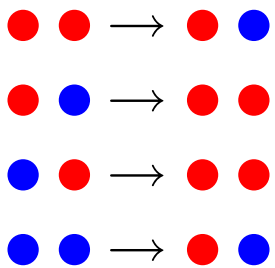


Exercice 0: When the number n of agents is fixed: continued

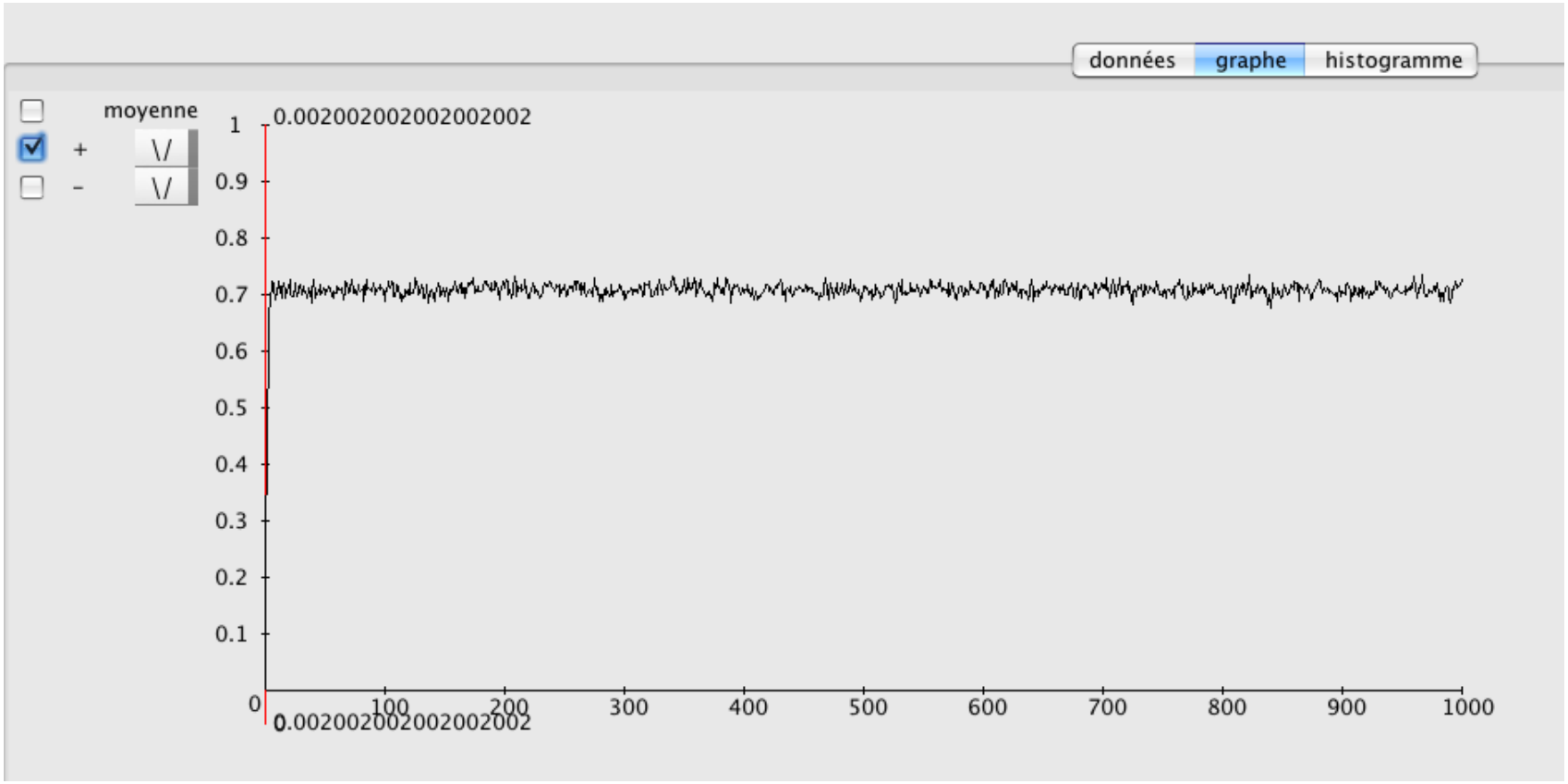
- By ergodic theorem, whatever the initial distribution is, the distribution of ●, and ●will converge
- to the unique stationary distribution of the Markov chain,
- an hence,

$E[p_{●}]$ will converge to some (computable) rational value.

When the number n of agents is large?



- What is the limit of $p_{\bullet}$ when n goes to infinity?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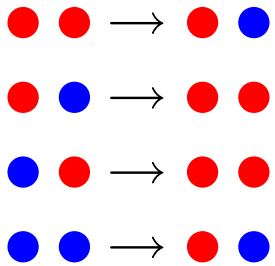
An informal approach



- The mean number of ● created,

$$\begin{aligned} b(p_{\bullet}) &= -1 * p_{\bullet}^2 + 1 * p_{\bullet}(1 - p_{\bullet}) + 1 * p_{\bullet}(1 - p_{\bullet}) + 1 * (1 - p_{\bullet})^2 \\ &= 1 - 2p_{\bullet}^2 \end{aligned}$$

- must be equal, at the limit to 0,
- and hence

$$p_{\bullet} = \frac{\sqrt{2}}{2},$$

at the limit.

Computing with large populations

- In other words,

  this protocol **computes** real number $\frac{\sqrt{2}}{2}$.

- Which real numbers can be computed in the above sense?

A scenario

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- Whenever two individuals meet, i.e., whenever they get within the transmission range of each-other, their respective mobile devices get connected, and potentially exchange data of common interest.
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Sub-menu

Large Population Protocols

- My favorite example
- The model
- Convergence to an ordinary differential equation
- Asymptotic Development
- Computing with large population protocols
- Our main result

General Case: Algorithm

An algorithm consists of

- a finite set of states $Q = \{q_1, q_2, \cdots, q_k\}$.
- transition rules, mapping pairs of states to pairs of states

Executions are given by:

- instantaneous configuration: $(x_1, x_2, \cdots, x_k) \in [0,1]^k$, with $\sum_i x_i = 1$.
- transitions between configuration: two agents are picked, and updated according to the rules of the algorithm.
- some states are marked.
- we want to consider the proportion of agents in marked states.

Remark: Algorithm are assumed independent of size of population!

Rescaling...

- Let $Y_i^{(n)}(t)$ be the random variables equal to the numbers of agents in state q_i at time t in a population of size n.
- Let

$$\bar{Y}_i^{(n)}(t) = \frac{1}{n} \cdot Y_i^{(n)}(t).$$

$$\bar{Y}^{(n)}(t) = (\bar{Y}_1^{(n)}(t), \ldots, \bar{Y}_{|Q|}^{(n)}(t)).$$

$$X^{(n)}(t) = \bar{Y}^{(n)}(\lfloor nt \rfloor) + (nt - \lfloor nt \rfloor) \cdot \left(\bar{Y}^{(n)}(\lfloor nt+1 \rfloor) - \bar{Y}^{(n)}(\lfloor nt \rfloor) \right).$$

- By definition,
  - $X^{(n)}(t)$ is a continuous-time Markov chain obtained by linear interpolation of $\bar{Y}^{(n)}$ with a time-acceleration of factor n.
  - After rescaling, the number of interactions occurring in one time unit is proportional to the number of agents in the population.

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Convergence to an ordinary differential equation

Proposition

For any initial conditions $Y^{(n)}(0)$ such that $Y^{(n)}(0) \to x$ when $n \to \infty$, the sequence of random processes $X^{(n)}$ converges in law to the solution of the stochastic differential equation (with degenerated brownian motion):

$$dX(t) = b(X(t))dt \quad \textit{with} \quad X(0) = x. \tag{1}$$

where

- $b : \mathbb{R}^{|Q|} \to \mathbb{R}^{|Q|}$ is the "balance" equation
  - Formally:

$$b(x) = \sum_{(q_1,q_2)\in Q^2} \left(x_{q_1} x_{q_2} \left(-e_{q_1} - e_{q_2} + \sum_{(q_3,q_4)\in Q^2} \delta_{q_1,q_2,q_3,q_4} (e_{q_3} + e_{q_4}) \right) \right) \tag{2}$$

where $\delta_{q_1,q_2,q_3,q_4} = 1$ if $q_1, q_2 \to q_3, q_4$, and 0 otherwise.

Sub-menu

Asymptotic Development

Theorem

Assume that the ordinary differential equation $dX(t)/dt = b(X(t))$ *has a (exponentialy) stable equilibrium* $b(x^*) = 0$.
Then there exists a neighborhood of x^* *such that, whenever* $\bar{Y}^{(n)}(0)$ *belongs to this neighborhood, we have*

$$\bar{Y}^{(n)}(t) \approx x^* + \frac{1}{\sqrt{n}} \mathcal{N}(0, \chi)$$

when $t \to \infty$,
where $\mathcal{N}(0, \chi)$ *is the centered gaussian distribution with covariance matrix* χ, *for some* χ.

Sub-menu

Computation

A real number ν is said to be computable if

- there exists a LPP $\mathcal{P}$, admitting finitely many equilibria,
- some of these finitely many equilibria $x^* = (x_1, x_2, ..., x_k) \in [0,1]^k$ is stable
- and $\sum_{q_i \in Q^+} x_i = \nu$ where Q^+ is the set of marked states for $\mathcal{P}$.

Sub-menu

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Main result

Theorem

ν *is computable by an LPP*

if and only if

$\nu \in [0, 1]$ *is algebraic.*

Sketch of proof

- Direct sense:
  - A computable number ν must correspond to a x^* with $b(x^*) = 0$.
  - b componentwise polynomial implies ν algebraic.
- Undirect sense:
  - All rational numbers are computable.
  - Derandomization: probabilistic large population protocols can be simulated by large population protocols.
  - Constructing equilibria.
  - Enforcing stability.

Menu

Conclusion

- We proposed a formalization of population protocols with large populations.
- We proved convergence in the general case.
- We gave an asymptotic development of the convergence in some cases.
- We characterize all computable reals numbers to be precisely algebraic numbers of $[0, 1]$.

Current and future work

- Variants on the model.

- Going further:
  - We characterize computable numbers.
  - Can we characterize computable predicates?

Sub-menu

Sketch of proof

Direct sense

Lemma

For every $\nu \in [0,1]$, if ν is computable by a LPP then ν is algebraic.

- a computable number must correspond to an equilibrium x^* of b: i.e. $b(x^*) = 0$.
- b is componentwise polynomial
- any solution of a system of polynomial equations which has finitely many solutions is component-wise algebraic.

back

Undirect sense: step 1
Computing all rationals

Lemma

Let $\nu \in [0,1]$ be a rational number. There exists a LPP computing ν.

- To compute $1/k$, consider the protocol $\mathcal{M}$ over states $Q_k = \{1, \ldots, k\}$ given by the following transitions rules:

$$i\ j \to (i+1)\ (j+1)$$

.

- To compute $p/q \in \mathbb{Q}$, set marked states in an appropriate way.

back

Undirect sense: step 2
Derandomization

We focus on probabilistic LPPs, where the transition rules are defined by

$$q_i \; q_j \longrightarrow \alpha_{i,j,k,l} \; q_k \; q_l$$

where all $\alpha_{i,j,k,l}$ are rational numbers.

Lemma

Let $\nu \in [0,1]$, and assume that there exists a probabilistic LPP computing ν, with rational probabilities. Then there exists a (deterministic) LPP computing ν.

◂ back

Undirect sense: step 3
Constructing Equilibria

Lemma

For any algebraic number $\nu \in [0,1]$, *there exists* $\delta \in \mathbb{N}$, $\lambda \in \mathbb{Q}$, *and a protocol* $\mathcal{P}$ *such that*
$(\nu, \lambda\nu^2, \lambda^2\nu^3, \ldots, \lambda^{\delta-2}\nu^{\delta-1}, 1 - \sum_{i=1}^{\delta-1} \lambda^{i-1}\nu^i)$ *is an equilibrium of* $\mathcal{P}$.

Undirect sense: step 3
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- Let $\nu \in (0,1]$ be an algebraic number, and let $P(X) = \sum_{i=0}^{\delta} a_i X^i$, be a polynomial such that $P(\nu) = 0$, and $P(0) > 0$.
- Build $\mathcal{P}_\epsilon$ with

$$\begin{aligned} dx_1 &= \epsilon(a_0 + a_1 x_1 + \sum_{i=2}^{\delta-1} \frac{a_{i+1}}{\lambda^{i-1}} x_{i-1} x_1) \\ dx_i &= \lambda x_1 x_{i-1} - x_i \text{ for every } i = 2, \ldots, \delta - 1 \\ dx_\delta &= -\sum_{i=1}^{\delta-1} dx_i. \end{aligned}$$

where $\lambda > 0$, and $\sum_{i=1}^{\delta-1} \lambda^{i-1}\nu^i \leq 1$.

Undirect sense: step 4
Enforcing stability

Lemma

Every algebraic number $\nu \in [0,1]$ is computable by an LPP.

back

Undirect sense: step 4
Enforcing stability

Lemma

Every algebraic number $\nu \in [0, 1]$ is computable by an LPP.

- Establish stability using Routh-Hurwitz criterion...
- by choosing ϵ sufficiently small.

back