Learning in Nonlinear Games

P. Mertikopoulos

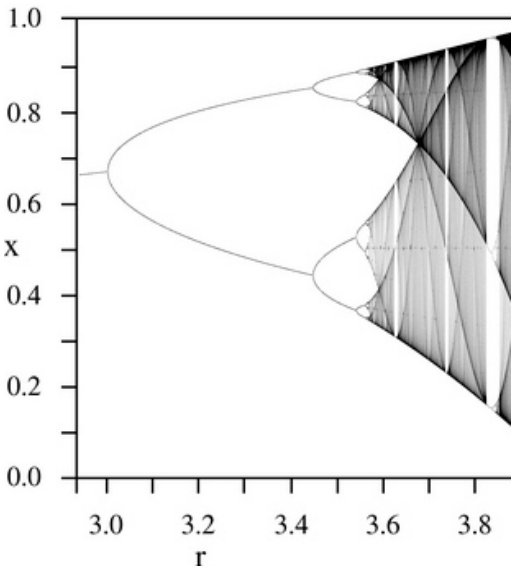
École Polytechnique, Department of Economics

AlgoGT 2011 – June 20, 2011

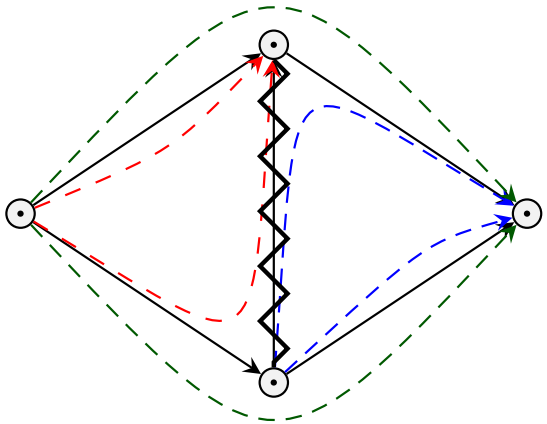
Motivation

Many game-like interactions have highly nonlinear utilities:

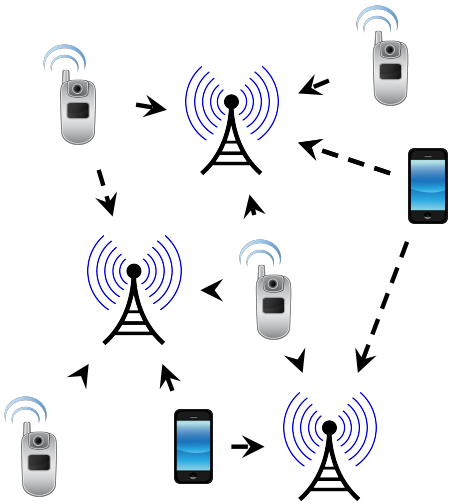
Population Growth Models



Congestion Models



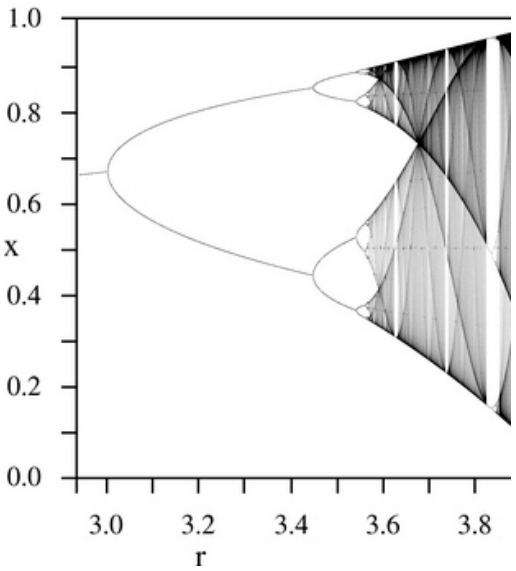
Wireless Networks



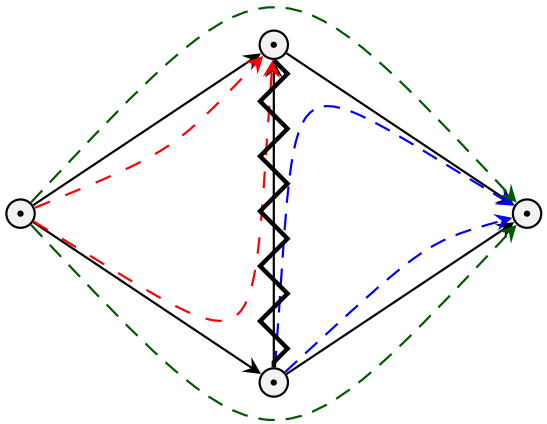
Motivation

Many game-like interactions have highly nonlinear utilities:

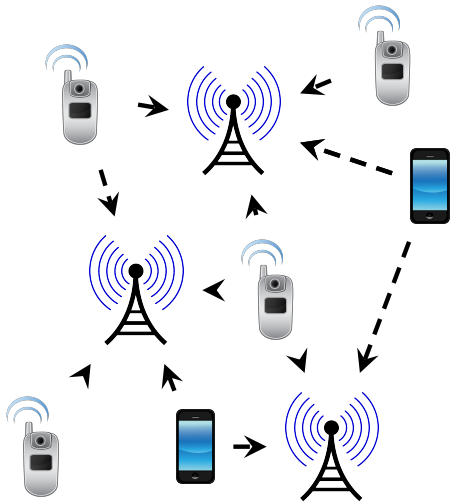
Population Growth Models



Congestion Models



Wireless Networks



- Learning methods usually work with finite, affine or convex structures.
- Can we control completely nonlinear games?

ÉCOLE POLYTECHNIQUE

Outline

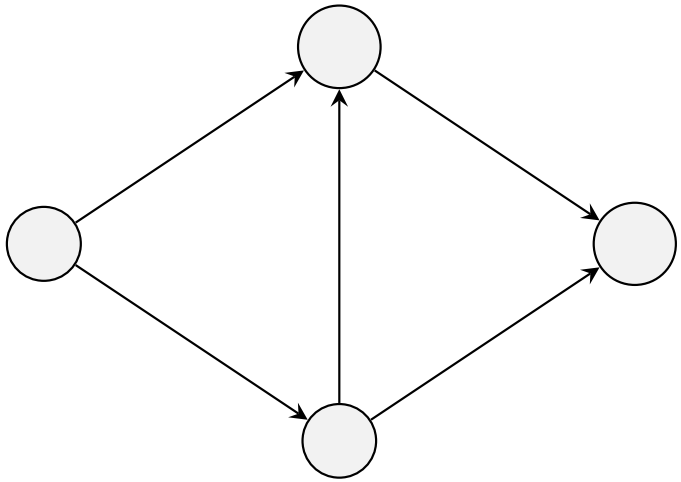
Simplicial State Spaces and Nonlinear Utilities I: Routing

Large-scale (nonatomic) congestion models consist of:

Simplicial State Spaces and Nonlinear Utilities I: Routing

Large-scale (nonatomic) congestion models consist of:

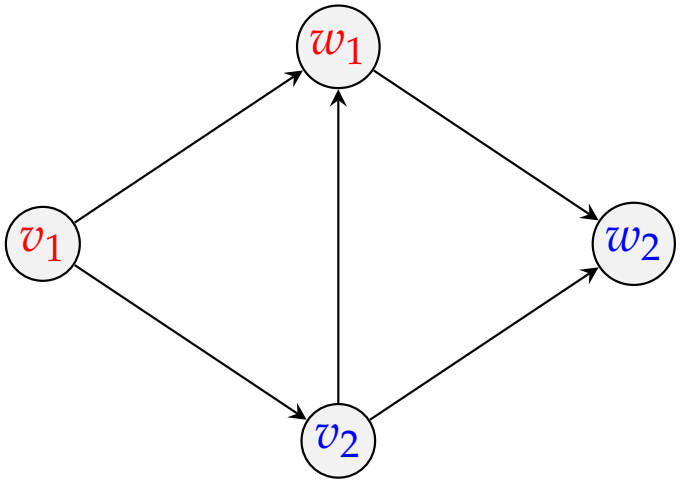
- An arbitrary directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$.



Simplicial State Spaces and Nonlinear Utilities I: Routing

Large-scale (nonatomic) congestion models consist of:

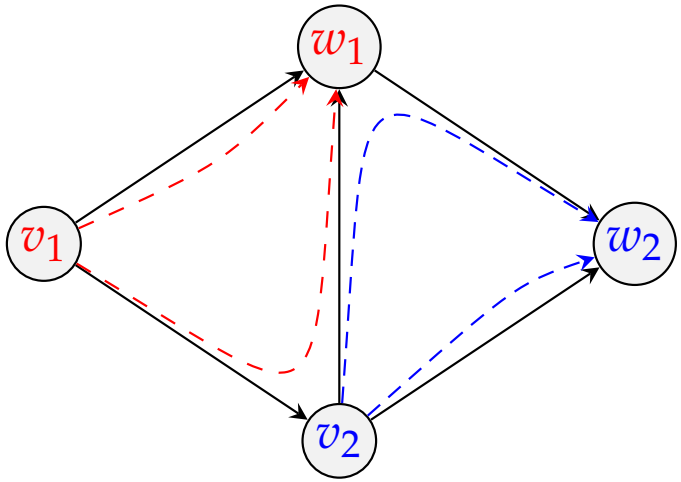
- An arbitrary directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$.
- A collection of origin-destination pairs $\mathcal{K} = \{1, \ldots, N\}$ with traffic rates ρ_k.



Simplicial State Spaces and Nonlinear Utilities I: Routing

Large-scale (nonatomic) congestion models consist of:

- An arbitrary directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$.
- A collection of origin-destination pairs $\mathcal{K} = \{1, \ldots, N\}$ with traffic rates ρ_k.
- A collection of paths $\mathcal{A}_k$ joining each pair.

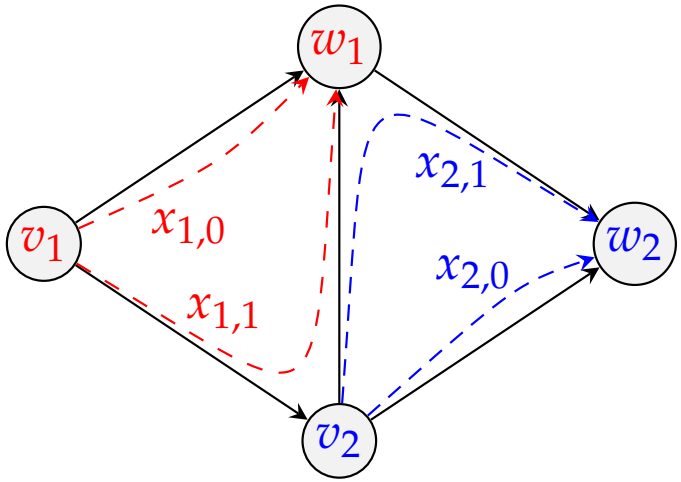


Simplicial State Spaces and Nonlinear Utilities I: Routing

Large-scale (nonatomic) congestion models consist of:

- An arbitrary directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$.
- A collection of origin-destination pairs $\mathcal{K} = \{1, \ldots, N\}$ with traffic rates ρ_k.
- A collection of paths $\mathcal{A}_k$ joining each pair.
- The space of each pair's traffic flows:

$$\Delta_k = \{x_k \in \mathbb{R}^{\mathcal{A}_k} : x_{k\alpha} \geq 0 \text{ and } \textstyle\sum_\alpha^k x_{k\alpha} = \rho_k\}.$$



Simplicial State Spaces and Nonlinear Utilities I: Routing

Large-scale (nonatomic) congestion models consist of:

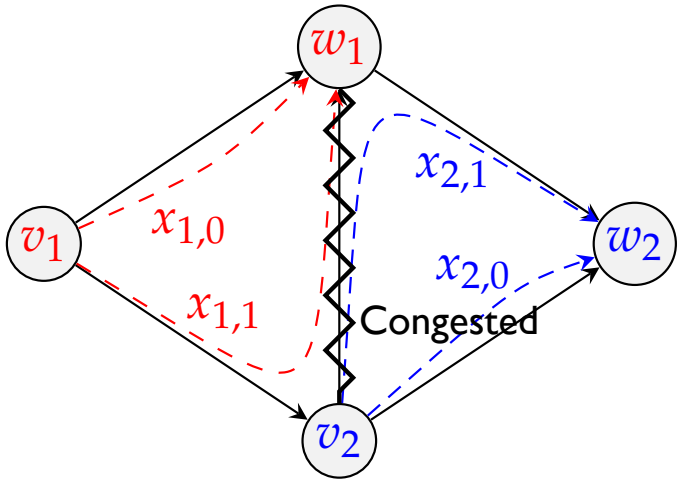
- An arbitrary directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$.
- A collection of origin-destination pairs $\mathcal{K} = \{1, \ldots, N\}$ with traffic rates ρ_k.
- A collection of paths $\mathcal{A}_k$ joining each pair.
- The space of each pair's traffic flows:

  $\Delta_k = \{x_k \in \mathbb{R}^{\mathcal{A}_k} : x_{k\alpha} \geq 0 \text{ and } \sum_\alpha^k x_{k\alpha} = \rho_k\}.$

- The network's latency functions:

  $$\omega_{k\alpha}(x) \equiv \sum_{r \in \alpha} \phi_r(y_r)$$

  where $y_r = \sum_{\alpha \ni r} x_{k\alpha}$ is the load on link r and ϕ_r is the expected delay.



$$\omega_{1,1}(x) = \phi_2(x_{1,1}) + \phi_5(x_{1,1} + x_{2,1}) \ldots$$

Simplicial State Spaces and Nonlinear Utilities I: Routing

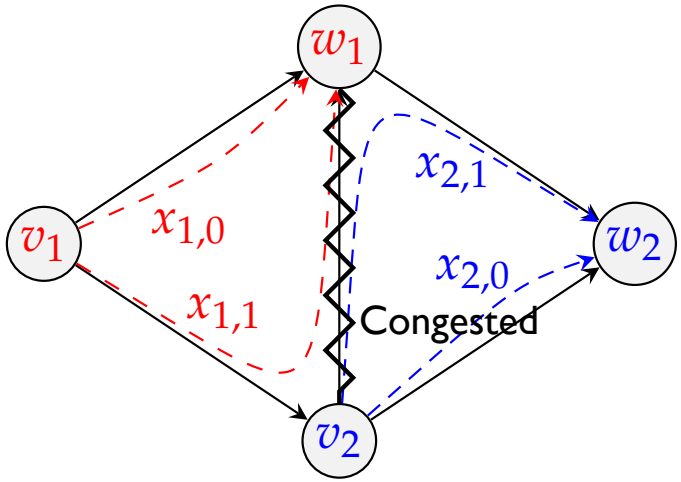
Large-scale (nonatomic) congestion models consist of:

- An arbitrary directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$.
- A collection of origin-destination pairs $\mathcal{K} = \{1, \ldots, N\}$ with traffic rates ρ_k.
- A collection of paths $\mathcal{A}_k$ joining each pair.
- The space of each pair's traffic flows:

  $\Delta_k = \{x_k \in \mathbb{R}^{\mathcal{A}_k} : x_{k\alpha} \geq 0 \text{ and } \sum_\alpha^k x_{k\alpha} = \rho_k\}$.
- The network's latency functions:

  $$\omega_{k\alpha}(x) \equiv \sum_{r \in \alpha} \phi_r(y_r)$$

  where $y_r = \sum_{\alpha \ni r} x_{k\alpha}$ is the load on link r and ϕ_r is the expected delay.



$$\omega_{1,1}(x) = \phi_2(x_{1,1}) + \phi_5(x_{1,1} + x_{2,1}) \ldots$$

Assumption O/D latencies $\omega_k(x) = \sum_\alpha^k x_{k\alpha}\omega_{k\alpha}(x)$ are increasing.

ÉCOLE POLYTECHNIQUE

Routing efficiency notions

Main objective: minimize delays. *But what does this really mean?*

Routing efficiency notions

Main objective: minimize delays. *But what does this really mean?*

- Optimum Distributions – traffic allocations which minimize the network's *aggregate latency* $\Omega(x) = \sum_{k,\alpha} x_{k\alpha}\omega_{k\alpha}(x)$:

$$\Omega(q) \leq \Omega(x) \text{ for all flow profiles } x \in \Delta.$$

Routing efficiency notions

Main objective: minimize delays. *But what does this really mean?*

- Optimum Distributions – traffic allocations which minimize the network's *aggregate latency* $\Omega(x) = \sum_{k,\alpha} x_{k\alpha}\omega_{k\alpha}(x)$:

$$\Omega(q) \leq \Omega(x) \text{ for all flow profiles } x \in \Delta.$$

- Nash/Wardrop Equilibrium – there are no profitable *unilateral* deviations:

$$\omega_{k\alpha}(q) \leq \omega_{k\beta}(q) \text{ for all } \alpha, \beta \in \mathcal{A}_k \text{ such that } q_{k\alpha} > 0.$$

"The journey times in all routes actually used are equal and less than those which would be experienced by a single vehicle on any unused route." (J. G. Wardrop, 1952)

Routing efficiency notions

Main objective: minimize delays. *But what does this really mean?*

- Optimum Distributions – traffic allocations which minimize the network's *aggregate latency* $\Omega(x) = \sum_{k,\alpha} x_{k\alpha}\omega_{k\alpha}(x)$:

$$\Omega(q) \leq \Omega(x) \text{ for all flow profiles } x \in \Delta.$$

- Nash/Wardrop Equilibrium – there are no profitable *unilateral* deviations:

$$\omega_{k\alpha}(q) \leq \omega_{k\beta}(q) \text{ for all } \alpha, \beta \in \mathcal{A}_k \text{ such that } q_{k\alpha} > 0.$$

"The journey times in all routes actually used are equal and less than those which would be experienced by a single vehicle on any unused route." (J. G. Wardrop, 1952)

These two notions are dual to each other:

	Minimize	Equilibrate
Optimum	$\sum_r y_r \phi_r(y_r)$	$\phi_r(y_r) + y_r \phi_r'(y_r)$
Wardrop	$\sum_r \int_0^{y_r} \phi_r(w)\, dw$	$\phi_r(y_r)$

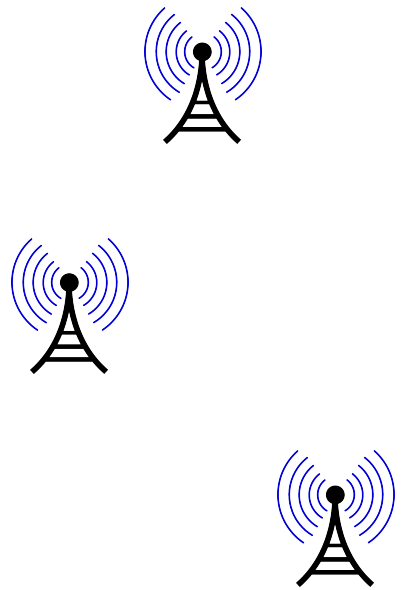
Semilinear Games II: Power allocation

For a different example, consider the following wireless network model:

Semilinear Games II: Power allocation

For a different example, consider the following wireless network model:

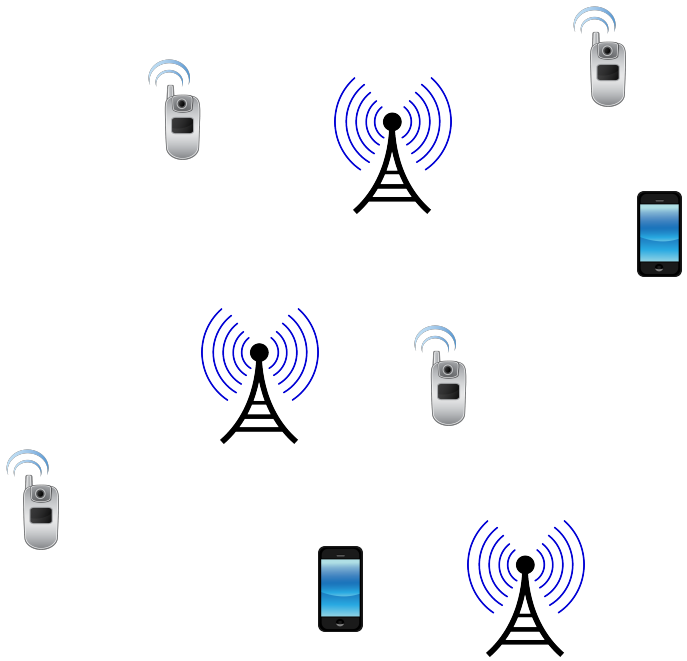
- A set of receivers $\mathcal{A} = \{1, \ldots, A\}$ with non-overlapping channels.



Semilinear Games II: Power allocation

For a different example, consider the following wireless network model:

- A set of receivers $\mathcal{A} = \{1, \ldots, A\}$ with non-overlapping channels.
- A set of users $\mathcal{K} = \{1, \ldots, K\}$ who transmit over multiple bands.

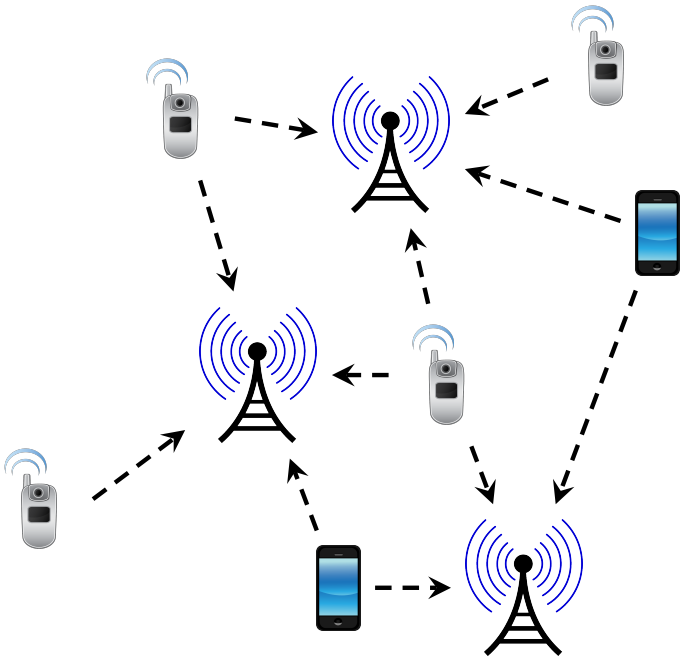


Semilinear Games II: Power allocation

For a different example, consider the following wireless network model:

- A set of receivers $\mathcal{A} = \{1, \ldots, A\}$ with non-overlapping channels.
- A set of users $\mathcal{K} = \{1, \ldots, K\}$ who transmit over multiple bands.
- Users split power among nodes:

$$p_{k\alpha} \geq 0, \quad \sum_{\alpha}^{k} p_{k\alpha} \leq P_k.$$



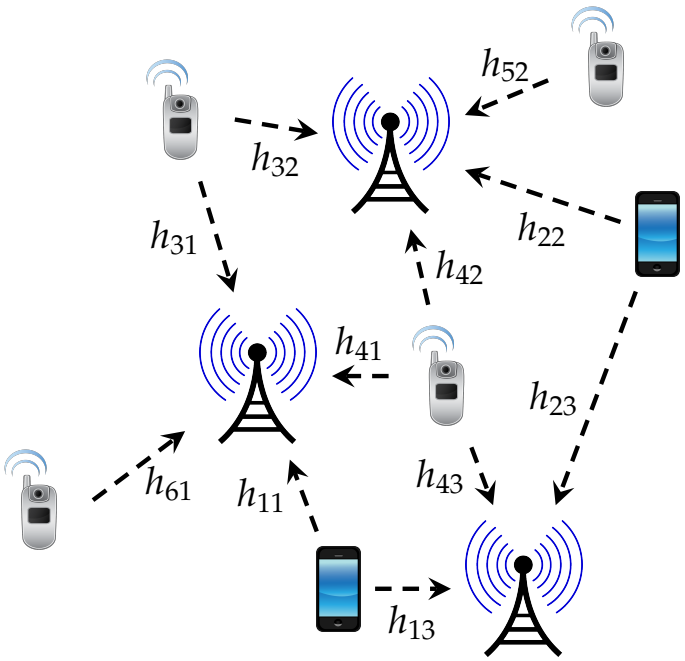
Semilinear Games II: Power allocation

For a different example, consider the following wireless network model:

- A set of receivers $\mathcal{A} = \{1, \ldots, A\}$ with non-overlapping channels.
- A set of users $\mathcal{K} = \{1, \ldots, K\}$ who transmit over multiple bands.
- Users split power among nodes:

$$p_{k\alpha} \geq 0, \quad \sum_{\alpha}^{k} p_{k\alpha} \leq P_k.$$

- A popular performance metric is the users' spectral efficiency:



$$u_k(p) = \sum_{\alpha \in \mathcal{A}} \log\left(1 + \mathrm{sinr}_{k\alpha}(p)\right) = \sum_{\alpha \in \mathcal{A}} \log\left(1 + \frac{g_{k\alpha} p_{k\alpha}}{\sigma_\alpha^2 + \sum_{\ell \neq k} g_{\ell\alpha} p_{\ell\alpha}}\right)$$

where $g_{k\alpha} = |h_{k\alpha}|^2$ are the channel coefficients and σ_α^2 is the noise.

Bird's Eye View of Semilinear Models

Similarities between these two games:

- The state space is a product of simplices $\Delta = \prod_k \Delta_k$.
- The users' utilities are nonlinear over Δ.
- Both games admit a potential function.

Bird's Eye View of Semilinear Models

Similarities between these two games:

- The state space is a product of simplices $\Delta = \prod_k \Delta_k$.
- The users' utilities are nonlinear over Δ.
- Both games admit a potential function.

Differences:

- The games are potential in different ways (Sandholm v. Monderer-Shapley)
- Different equilibrial structure (but both at the minimum of the potential)
- State-specific v. undecomposable utilities:
  - $u_k(x) = \sum_\alpha^k x_{k\alpha} u_{k\alpha}(x)$ in the routing model
  - no such meaningful decomposition in the wireless network model

ÉCOLE POLYTECHNIQUE

Complexity of Calculating Efficient States

Unfortunately, efficient states are not very easy to calculate:

- Nonlinear convex problem with exponentially many variables.
- Proposed algorithms (e.g. flow deviation or water-filling) have important drawbacks:
  - Computationally intensive (e.g. solve MFDL at each step)
  - Not wholly distributed (must perform a global linear search)
  - Reduced convergence rates: search directions quickly become suboptimal

Complexity of Calculating Efficient States

Unfortunately, efficient states are not very easy to calculate:

- Nonlinear convex problem with exponentially many variables.
- Proposed algorithms (e.g. flow deviation or water-filling) have important drawbacks:
  - Computationally intensive (e.g. solve MFDL at each step)
  - Not wholly distributed (must perform a global linear search)
  - Reduced convergence rates: search directions quickly become suboptimal

It is thus natural to apply ideas from distributed learning in games:

- Agents only possess and process a limited amount of information.
- No exogenous forcing needed: evolution is steered by individual objectives.

Complexity of Calculating Efficient States

Unfortunately, efficient states are not very easy to calculate:

- Nonlinear convex problem with exponentially many variables.
- Proposed algorithms (e.g. flow deviation or water-filling) have important drawbacks:
  - Computationally intensive (e.g. solve MFDL at each step)
  - Not wholly distributed (must perform a global linear search)
  - Reduced convergence rates: search directions quickly become suboptimal

It is thus natural to apply ideas from distributed learning in games:

- Agents only possess and process a limited amount of information.
- No exogenous forcing needed: evolution is steered by individual objectives.

The catch: the learning process must actually work!

Exponential Learning

A scheme which fits the bill particularly well is that of "exponential learning":

Exponential Learning

A scheme which fits the bill particularly well is that of "exponential learning":

1. Take the agents' marginal utilities:

$$v_{k\alpha}(x) = -\frac{\partial \Phi}{\partial x_{k\alpha}}$$

(which conincide for both decomposable and state-specific utilities)

Exponential Learning

A scheme which fits the bill particularly well is that of "exponential learning":

1. Take the agents' marginal utilities:

$$v_{k\alpha}(x) = -\frac{\partial \Phi}{\partial x_{k\alpha}}$$

(which conincide for both decomposable and state-specific utilities)

2. Agents record their choices' performance:

$$V_{k\alpha}(t + dt) = V_{k\alpha}(t) + v_{k\alpha}(x(t))\, dt$$

Exponential Learning

A scheme which fits the bill particularly well is that of "exponential learning":

1. Take the agents' marginal utilities:

$$v_{k\alpha}(x) = -\frac{\partial \Phi}{\partial x_{k\alpha}}$$

(which conincide for both decomposable and state-specific utilities)

2. Agents record their choices' performance:

$$V_{k\alpha}(t + dt) = V_{k\alpha}(t) + v_{k\alpha}(x(t))\, dt$$

3. State variables are updated according to the Gibbs distribution:

$$x_{k\alpha}(t) = \frac{e^{\lambda_k V_{k\alpha}(t)}}{\sum_\beta e^{\lambda_k V_{k\beta}(t)}},$$

where the "learning parameters" λ_k are player-specific inverse temperatures

Exponential Learning

A scheme which fits the bill particularly well is that of "exponential learning":

1. Take the agents' marginal utilities:

$$v_{k\alpha}(x) = -\frac{\partial \Phi}{\partial x_{k\alpha}}$$

(which conincide for both decomposable and state-specific utilities)

2. Agents record their choices' performance:

$$V_{k\alpha}(t + dt) = V_{k\alpha}(t) + v_{k\alpha}(x(t))\, dt$$

3. State variables are updated according to the Gibbs distribution:

$$x_{k\alpha}(t) = \frac{e^{\lambda_k V_{k\alpha}(t)}}{\sum_\beta e^{\lambda_k V_{k\beta}(t)}},$$

where the "learning parameters" λ_k are player-specific inverse temperatures

4. Decouple to obtain the (rate-adjusted) replicator dynamics:

$$\frac{dx_{k\alpha}}{dt} = \lambda_k x_{k\alpha} \left(v_{k\alpha}(x) - \sum_\beta^k x_{k\beta} v_k(x) \right)$$

ÉCOLE POLYTECHNIQUE

Convergence Properties of the Replicator Dynamics

So, how well does replicator learning fare?

Convergence Properties of the Replicator Dynamics

So, how well does replicator learning fare?

Theorem (Standard stuff)

The minimum sets of the potential are Lyapunov stable (and usually attracting). [Sandholm '01]

Convergence Properties of the Replicator Dynamics

So, how well does replicator learning fare?

Theorem (Standard stuff)

The minimum sets of the potential are Lyapunov stable (and usually attracting). [Sandholm '01]

Theorem (A bit better)

In stable games, interior solution trajectories always converge to a minimum point. [M '10]

Convergence Properties of the Replicator Dynamics

So, how well does replicator learning fare?

Theorem (Standard stuff)

The minimum sets of the potential are Lyapunov stable (and usually attracting). [Sandholm '01]

Theorem (A bit better)

In stable games, interior solution trajectories always converge to a minimum point. [M '10]

And for a more practical question: how fast is this convergence?

Convergence Properties of the Replicator Dynamics

So, how well does replicator learning fare?

Theorem (Standard stuff)

The minimum sets of the potential are Lyapunov stable (and usually attracting). [Sandholm '01]

Theorem (A bit better)

In stable games, interior solution trajectories always converge to a minimum point. [M '10]

And for a more practical question: how fast is this convergence?

Theorem (M 2010)

Let $x(t)$ be an interior solution orbit, and let $q = \lim_{t\to\infty} x(t)$. Then:

$$D_{KL}(q \,\|\, x(t)) \leq h_0 e^{-ct},$$

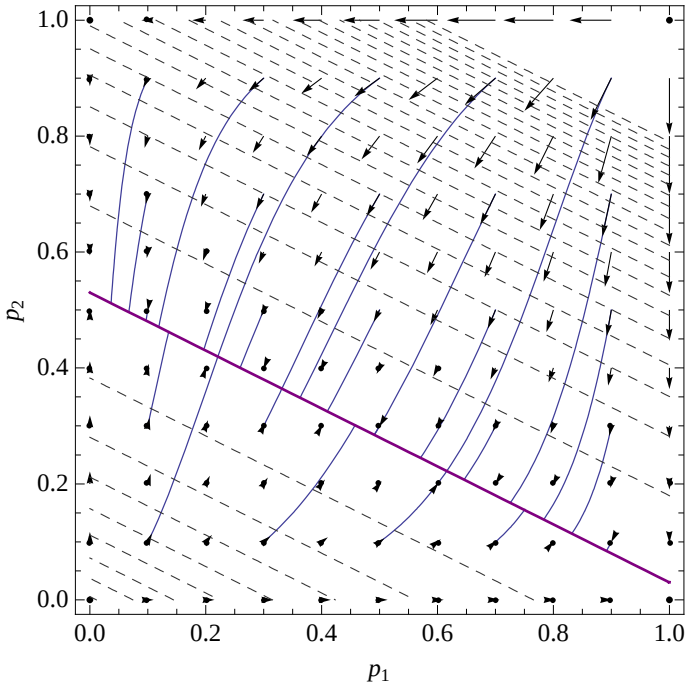
where D_{KL} denotes the Kullback-Leibler divergence, $h_0 = D_{KL}(q \,\|\, x(0))$ is the initial relative entropy, and c is a positive constant.

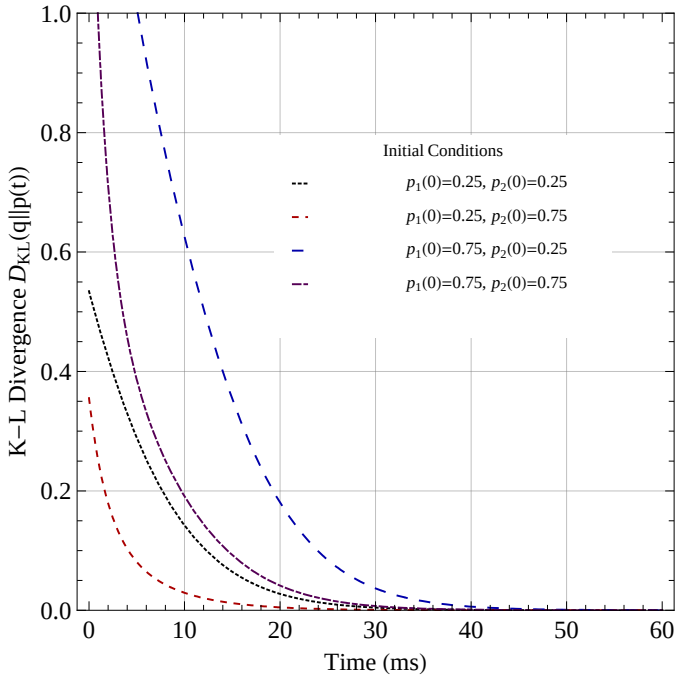
In other words, exponential learning equilibrates exponentially quickly:

solution orbits hit an ε-neighborhood of an optimum distribution in time $\mathcal{O}(\log(1/\varepsilon))$.

Convergence of the Replicator Dynamics

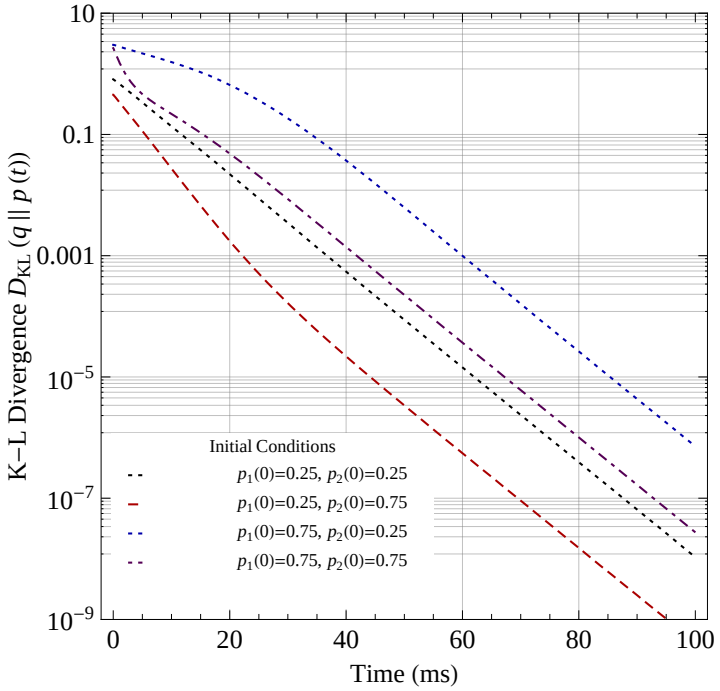
Thus, almost all initial conditions converge to an efficient state

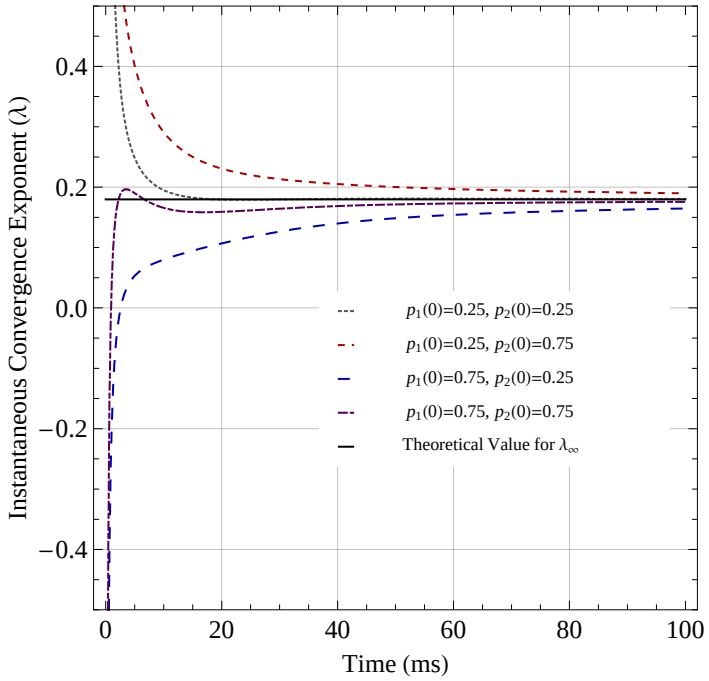




Convergence of the Replicator Dynamics

Thus, almost all initial conditions converge to an efficient state in time which is at most $\mathcal{O}(\log(1/\varepsilon))$.





ÉCOLE POLYTECHNIQUE

Outline

Beyond Simplices: an example

A natural extension of our wireless network example: multi-antenna systems.

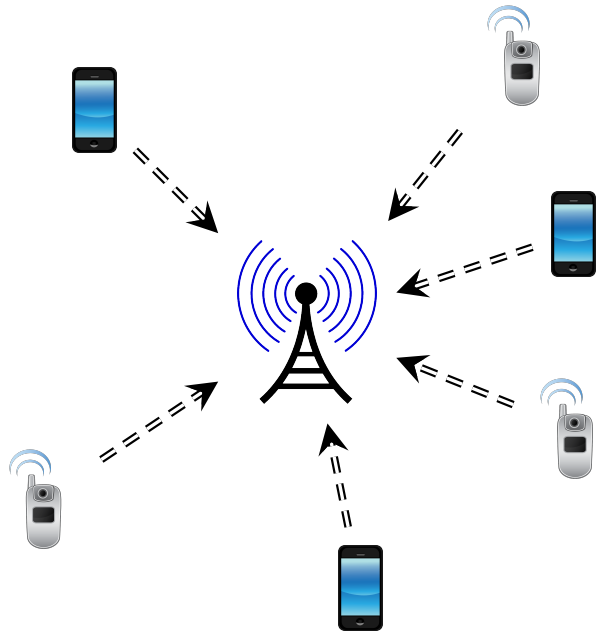
Beyond Simplices: an example

A natural extension of our wireless network example: multi-antenna systems.

- The received signal is:

$$\mathbf{y} = \sum_k \mathbf{H}_k \cdot \mathbf{x_k} + \mathbf{z}$$

where $\mathbf{x_k}$ is k's transmitted vector



Beyond Simplices: an example

A natural extension of our wireless network example: multi-antenna systems.

- The received signal is:

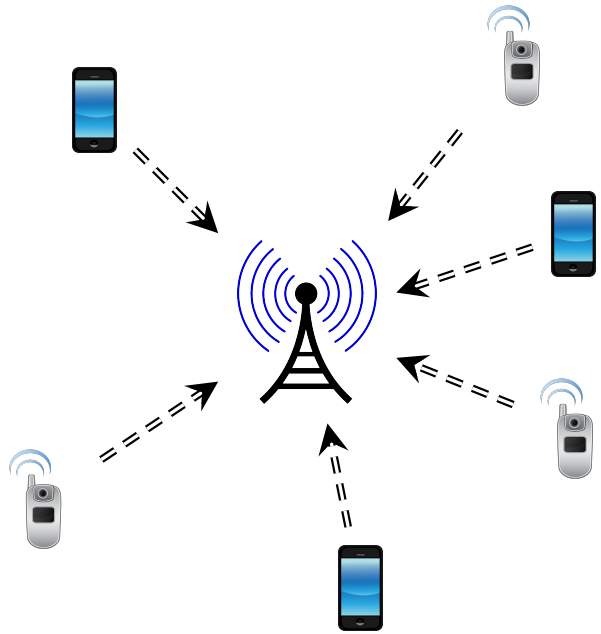
$$\mathbf{y} = \sum_k \mathbf{H}_k \cdot \mathbf{x_k} + \mathbf{z}$$

where $\mathbf{x_k}$ is k's transmitted vector

- Users choose a covariance matrix

$$\mathbf{Q}_k = \mathbf{E}\left[\mathbf{x}_k \cdot \mathbf{x}_k^\dagger\right]$$

(single antenna $\Longrightarrow$ diagonal $\mathbf{Q}$)



Beyond Simplices: an example

A natural extension of our wireless network example: multi-antenna systems.

- The received signal is:

$$\mathbf{y} = \sum_k \mathbf{H}_k \cdot \mathbf{x_k} + \mathbf{z}$$

where $\mathbf{x_k}$ is k's transmitted vector

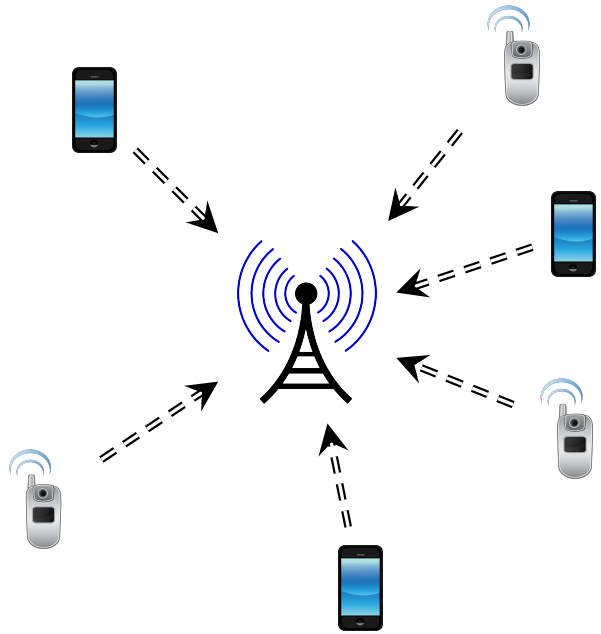
- Users choose a covariance matrix

$$\mathbf{Q}_k = \mathbf{E}\left[\mathbf{x}_k \cdot \mathbf{x}_k^\dagger\right]$$

(single antenna $\Longrightarrow$ diagonal $\mathbf{Q}$)

- Strategy spaces are now cones:

$$M_k = \{\mathbf{Q}_k : \mathbf{Q}_k \succeq 0, \mathrm{tr}(\mathbf{Q}_k) \leq P_k\}$$



Beyond Simplices: an example

A natural extension of our wireless network example: multi-antenna systems.

- The received signal is:

$$\mathbf{y} = \sum_k \mathbf{H}_k \cdot \mathbf{x_k} + \mathbf{z}$$

where $\mathbf{x_k}$ is k's transmitted vector

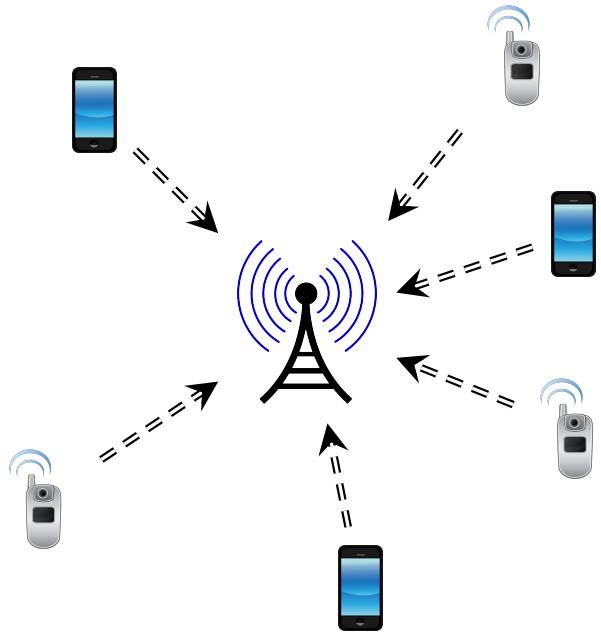
- Users choose a covariance matrix

$$\mathbf{Q}_k = \mathbf{E}\left[\mathbf{x}_k \cdot \mathbf{x}_k^\dagger\right]$$

(single antenna $\implies$ diagonal $\mathbf{Q}$)

- Strategy spaces are now cones:

$$M_k = \{\mathbf{Q}_k : \mathbf{Q}_k \succeq 0, \mathrm{tr}(\mathbf{Q}_k) \leq P_k\}$$



- Utility functions are undecomposable:

$$u_k(\mathbf{Q}) = \log\det\left(\mathbf{I} + \sum_k \mathbf{H_k Q_k H_k^\dagger}\right) - \log\det\left(\mathbf{I} + \sum_{\ell \neq k} \mathbf{H}_\ell \mathbf{Q}_\ell \mathbf{H}_\ell^\dagger\right)$$

How can we control this totally nonlinear (but still relevant) system?

Beyond Simplices: an example

The challenge: non-simplicial state space $\Longrightarrow$ doom for replicator dynamics ☹

Beyond Simplices: an example

The challenge: non-simplicial state space $\Longrightarrow$ doom for replicator dynamics ☹

Vanilla game-theoretic solution:

1. Discretize the state space.
2. Take (multi)linearized versions of the game's utilities.
3. Employ your favorite discrete learning algorithm.
4. Converge to a state which is efficient ``on average".

Downside: efficiency $\Longrightarrow$ fine discretization $\Longrightarrow$ high complexity

Beyond Simplices: an example

The challenge: non-simplicial state space $\Longrightarrow$ doom for replicator dynamics ☹

Vanilla game-theoretic solution:

1. Discretize the state space.
2. Take (multi)linearized versions of the game's utilities.
3. Employ your favorite discrete learning algorithm.
4. Converge to a state which is efficient ``on average".

Downside: efficiency $\Longrightarrow$ fine discretization $\Longrightarrow$ high complexity

``Distributed" optimization workaround: water-filling and its friends

+ Converges to equilibrium in many scenarios.
- High complexity.
- Not entirely distributed.

Beyond Simplices: an example

Still, can we salvage replicator-type approaches?

- Can we write a dynamical system which respects the game's state space?
- Can we obtain good convergence properties?

Beyond Simplices: an example

Still, can we salvage replicator-type approaches?

- Can we write a dynamical system which respects the game's state space?
- Can we obtain good convergence properties?

Yes, we can!

$$\frac{d\mathbf{Q}_k}{dt} = \mathbf{Q}_k \cdot \left(\mathbf{V}_k - \operatorname{tr}(\mathbf{Q}_k \mathbf{V}_k) / \operatorname{tr}(\mathbf{Q}_k) \mathbf{I} \right)$$

where $\mathbf{V}_k = -\frac{\partial \Phi}{\partial \mathbf{Q}_k}$ is the matrix derivative of the game's potential w.r.t. $\mathbf{Q}_k$.

(Diagonal $\mathbf{Q}_k$'s reduce to the standard replicator dynamics ✓)

Beyond Simplices: an example

Still, can we salvage replicator-type approaches?

- Can we write a dynamical system which respects the game's state space?
- Can we obtain good convergence properties?

Yes, we can!

$$\frac{d\mathbf{Q}_k}{dt} = \mathbf{Q}_k \cdot \left(\mathbf{V}_k - \operatorname{tr}(\mathbf{Q}_k \mathbf{V}_k) / \operatorname{tr}(\mathbf{Q}_k) \mathbf{I} \right)$$

where $\mathbf{V}_k = -\frac{\partial \Phi}{\partial \mathbf{Q}_k}$ is the matrix derivative of the game's potential w.r.t. $\mathbf{Q}_k$.

(Diagonal $\mathbf{Q}_k$'s reduce to the standard replicator dynamics ✓)

Theorem (M-Moustakas '11)

1. *The matrix replicator dynamics leave the game's state space invariant.*
2. *Solution trajectories converge to equilibrium.*

Nonlinear Learning — Non-convex Optimization

General optimization problem formulation:

- Let M be a smooth manifold with boundary embedded in $\mathbb{R}^n$:

$$M = \{x \in \mathbb{R}^n : f_k(x) = 0,\, g_r(x) \geq 0,\, k = 1, \ldots, K,\, r = 1, \ldots, R\}$$

- Let $\phi \colon \mathbb{R}^n \to \mathbb{R}$ be a smooth function.

Nonlinear Learning — Non-convex Optimization

General optimization problem formulation:

- Let M be a smooth manifold with boundary embedded in $\mathbb{R}^n$:

$$M = \{x \in \mathbb{R}^n : f_k(x) = 0,\, g_r(x) \geq 0,\, k = 1, \ldots, K,\, r = 1, \ldots, R\}$$

- Let $\phi \colon \mathbb{R}^n \to \mathbb{R}$ be a smooth function.

The challenge: find a smooth vector field $V \equiv V(\phi)$ on M such that the dynamical system $\dot{x} = -V(x)$ converges to the minimum points of ϕ.

Nonlinear Learning — Non-convex Optimization

General optimization problem formulation:

- Let M be a smooth manifold with boundary embedded in $\mathbb{R}^n$:

$$M = \{x \in \mathbb{R}^n : f_k(x) = 0,\, g_r(x) \geq 0,\, k = 1, \ldots, K,\, r = 1, \ldots, R\}$$

- Let $\phi \colon \mathbb{R}^n \to \mathbb{R}$ be a smooth function.

The challenge: find a smooth vector field $V \equiv V(\phi)$ on M such that the dynamical system $\dot{x} = -V(x)$ converges to the minimum points of ϕ.

Easy if M is boundaryless ($V = \operatorname{grad}(\phi|_M)$ works!), but what happens if M does have a boundary (viz. power/rate constraints of previous examples)?

The General Idea

Guidelines and desiderata:

- Project $\mathrm{grad}(\phi)$ on M.
- ``Temper" $\mathrm{grad}(\phi)$ to avoid collisions with the boundary.
- Recover standard dynamics (think replicator) in simple cases.

A First Implementation

A promising approach is the following:

1. Let $w_k = \mathrm{grad}(f_k)$ be the ``outward normals" of M.

A First Implementation

A promising approach is the following:

1. Let $w_k = \mathrm{grad}(f_k)$ be the ``outward normals" of M.
2. Take the ``normal projection" of the objective ϕ:

$$\mathrm{grad}_\perp(\phi) \equiv \mathrm{proj}_W \mathrm{grad}(\phi) = \sum_k \frac{\langle w_k, \mathrm{grad}(\phi)\rangle}{\|w_k\|^2} w_k$$

A First Implementation

A promising approach is the following:

1. Let $w_k = \mathrm{grad}(f_k)$ be the ``outward normals" of M.
2. Take the ``normal projection" of the objective ϕ:

$$\mathrm{grad}_\perp(\phi) \equiv \mathrm{proj}_W \mathrm{grad}(\phi) = \sum_k \frac{\langle w_k, \mathrm{grad}(\phi)\rangle}{\|w_k\|^2} w_k$$

3. Let $\mathrm{grad}_M(\phi) = \mathrm{grad}(\phi) - \mathrm{grad}_\perp(\phi)$ be the projection of $\mathrm{grad}(\phi)$ on M.

A First Implementation

A promising approach is the following:

1. Let $w_k = \mathrm{grad}(f_k)$ be the ``outward normals" of M.
2. Take the ``normal projection" of the objective ϕ:

$$\mathrm{grad}_\perp(\phi) \equiv \mathrm{proj}_W \mathrm{grad}(\phi) = \sum_k \frac{\langle w_k, \mathrm{grad}(\phi)\rangle}{\|w_k\|^2} w_k$$

3. Let $\mathrm{grad}_M(\phi) = \mathrm{grad}(\phi) - \mathrm{grad}_\perp(\phi)$ be the projection of $\mathrm{grad}(\phi)$ on M.
4. Temper $\widetilde{V} \equiv \mathrm{grad}_M(\phi)$ by setting $V_\alpha = g_\alpha \widetilde{V}_\alpha$ where

$$g_\alpha(x) = \prod_{\alpha \in r} g_r(x)$$

and ``$\alpha \in r$" here means that $\frac{\partial g_r}{\partial x_\alpha} \neq 0$.

A First Implementation

A promising approach is the following:

1. Let $w_k = \mathrm{grad}(f_k)$ be the ``outward normals" of M.
2. Take the ``normal projection" of the objective ϕ:

$$\mathrm{grad}_\perp(\phi) \equiv \mathrm{proj}_W \mathrm{grad}(\phi) = \sum_k \frac{\langle w_k, \mathrm{grad}(\phi)\rangle}{\|w_k\|^2} w_k$$

3. Let $\mathrm{grad}_M(\phi) = \mathrm{grad}(\phi) - \mathrm{grad}_\perp(\phi)$ be the projection of $\mathrm{grad}(\phi)$ on M.
4. Temper $\widetilde{V} \equiv \mathrm{grad}_M(\phi)$ by setting $V_\alpha = g_\alpha \widetilde{V}_\alpha$ where

$$g_\alpha(x) = \prod_{\alpha \in r} g_r(x)$$

   and ``$\alpha \in r$" here means that $\frac{\partial g_r}{\partial x_\alpha} \neq 0$.

5. Learn by following the dynamics $\dot{x}_\alpha = -V_\alpha(x) = -g_\alpha(x)\,\mathrm{grad}_\alpha(\phi)$.

A Constructive Failure

This approach fails (but constructively so) even in the simplex case.

- $\phi\colon \mathbb{R}^n \to \mathbb{R}$ is a function to be minimized over the unit simplex.
- $f(x) \equiv \sum_\alpha x_\alpha - 1 = 0$ defines the boundaryless manifold.
- $g_\alpha(x) = x_\alpha \geq 0$, $\alpha = 1, \ldots, n$, defines the boundary.

A Constructive Failure

This approach fails (but constructively so) even in the simplex case.

- $\phi\colon \mathbb{R}^n \to \mathbb{R}$ is a function to be minimized over the unit simplex.
- $f(x) \equiv \sum_\alpha x_\alpha - 1 = 0$ defines the boundaryless manifold.
- $g_\alpha(x) = x_\alpha \geq 0$, $\alpha = 1, \ldots, n$, defines the boundary.

The outlined approach then gives:

$$\dot{x}_\alpha = -x_\alpha \left(\frac{\partial \phi}{\partial x_\alpha} - \frac{1}{n} \sum_\beta \frac{\partial \phi}{\partial x_\beta} \right)$$

which fails to remain in the unit simplex ($\sum_\alpha \dot{x}_\alpha \neq 0$).

Reason: the tempering meddles with the projection phase and vice versa.

Overcoming the Collision vs. Projection Conflict

The problem is subtle (but profound!): what does ``project" mean?

ÉCOLE POLYTECHNIQUE

Overcoming the Collision vs. Projection Conflict

The problem is subtle (but profound!): what does ``project" mean?

- The Euclidean metric is not natural!

Overcoming the Collision vs. Projection Conflict

The problem is subtle (but profound!): what does ``project" mean?

- The Euclidean metric is not natural!
- Instead, consider the ``complementary slackness" metric g:

$$\langle u, v \rangle = \sum_\alpha u_\alpha g_\alpha v_\alpha$$

where $g_\alpha = \prod_{\alpha \in r} g_r$ are the complementary slackness scale factors.

Overcoming the Collision vs. Projection Conflict

The problem is subtle (but profound!): what does ``project" mean?

- The Euclidean metric is not natural!
- Instead, consider the ``complementary slackness" metric g:

$$\langle u, v \rangle = \sum_\alpha u_\alpha g_\alpha v_\alpha$$

where $g_\alpha = \prod_{\alpha \in r} g_r$ are the complementary slackness scale factors.

- We then obtain the generalized replicator dynamics:

$$\dot{x}_\alpha = -\operatorname{proj}_\alpha \left(\operatorname{grad}_M(\phi) \right)$$

where the projection on the α-th coordinate is taken w.r.t. g and $\operatorname{grad}_M(\phi)) = \operatorname{grad}(\phi) - \operatorname{grad}_\perp(\phi)$ is the gradient of ϕ with respect to g.

Overcoming the Collision vs. Projection Conflict

The problem is subtle (but profound!): what does ``project" mean?

- The Euclidean metric is not natural!
- Instead, consider the ``complementary slackness" metric g:

$$\langle u, v \rangle = \sum_\alpha u_\alpha g_\alpha v_\alpha$$

where $g_\alpha = \prod_{\alpha \in r} g_r$ are the complementary slackness scale factors.

- We then obtain the generalized replicator dynamics:

$$\dot{x}_\alpha = -\operatorname{proj}_\alpha \left(\operatorname{grad}_M(\phi) \right)$$

where the projection on the α-th coordinate is taken w.r.t. g and $\operatorname{grad}_M(\phi)) = \operatorname{grad}(\phi) - \operatorname{grad}_\perp(\phi)$ is the gradient of ϕ with respect to g.

Theorem (M 2011)

If the embedding of M has no singular points, then:

- *The generalized replicator dynamics are always tangent to M.*
- *The objective function ϕ is (weakly) decreasing along solution orbits.*

Typical Results for the Generalized Replicator Dynamics

Hard part is over! A simple product allows us to learn in nonlinear games:

$$\dot{x}_k = -\operatorname{proj}_{M_k}\left(\operatorname{grad}_{M_k}(u_k)\right),$$

where M_k is the strategy manifold of agent k and $u_k\colon \prod_k M_k \to \mathbb{R}$ his utility.

Typical Results for the Generalized Replicator Dynamics

Hard part is over! A simple product allows us to learn in nonlinear games:

$$\dot{x}_k = -\operatorname{proj}_{M_k}\left(\operatorname{grad}_{M_k}(u_k)\right),$$

where M_k is the strategy manifold of agent k and $u_k \colon \prod_k M_k \to \mathbb{R}$ his utility.

Theorem

Let $\mathfrak{G} = (\mathcal{K}, M_k, u_k)$ be a nonlinear game. Then:

- *The stationary points of the generalized replicator dynamics are restricted equilibria (when they exist).*

Typical Results for the Generalized Replicator Dynamics

Hard part is over! A simple product allows us to learn in nonlinear games:

$$\dot{x}_k = -\operatorname{proj}_{M_k}\left(\operatorname{grad}_{M_k}(u_k)\right),$$

where M_k is the strategy manifold of agent k and $u_k\colon \prod_k M_k \to \mathbb{R}$ his utility.

Theorem

Let $\mathfrak{G} = (\mathcal{K}, M_k, u_k)$ be a nonlinear game. Then:

- *The stationary points of the generalized replicator dynamics are restricted equilibria (when they exist).*
- *If $x(t) \to q$, then x^* is a restricted equilibrium.*

Typical Results for the Generalized Replicator Dynamics

Hard part is over! A simple product allows us to learn in nonlinear games:

$$\dot{x}_k = -\operatorname{proj}_{M_k}\left(\operatorname{grad}_{M_k}(u_k)\right),$$

where M_k is the strategy manifold of agent k and $u_k\colon \prod_k M_k \to \mathbb{R}$ his utility.

Theorem

Let $\mathfrak{G} = (\mathcal{K}, M_k, u_k)$ be a nonlinear game. Then:

- *The stationary points of the generalized replicator dynamics are restricted equilibria (when they exist).*
- *If $x(t) \to q$, then x^* is a restricted equilibrium.*
- *If, in addition, the game admits a potential function ϕ, then all generic minima of ϕ (those that satisfy the strict KKT conditions) are asymptotically stable.*

Typical Results for the Generalized Replicator Dynamics

Hard part is over! A simple product allows us to learn in nonlinear games:

$$\dot{x}_k = -\operatorname{proj}_{M_k}\left(\operatorname{grad}_{M_k}(u_k)\right),$$

where M_k is the strategy manifold of agent k and $u_k \colon \prod_k M_k \to \mathbb{R}$ his utility.

Theorem

Let $\mathfrak{G} = (\mathcal{K}, M_k, u_k)$ be a nonlinear game. Then:

- *The stationary points of the generalized replicator dynamics are restricted equilibria (when they exist).*
- *If $x(t) \to q$, then x^* is a restricted equilibrium.*
- *If, in addition, the game admits a potential function ϕ, then all generic minima of ϕ (those that satisfy the strict KKT conditions) are asymptotically stable.*
- $\ldots$

ÉCOLE POLYTECHNIQUE

Conclusions and Future Directions

Some conclusions:

- Many relevant problems (esp. in engineering and OR) call for thinking outside the box of finite/population games.
- Existing results: mostly customized ad-hoc solutions to specific problems.
- Generalized replicator dynamics: first step in developing

Conclusions and Future Directions

Some conclusions:

- Many relevant problems (esp. in engineering and OR) call for thinking outside the box of finite/population games.
- Existing results: mostly customized ad-hoc solutions to specific problems.
- Generalized replicator dynamics: first step in developing

And some future directions:

- Statibility in non-potential games
- Rates of convergence (usually epxponential, but…)
- Generalize other important dynamics (BR, BNN, etc.)
- . . .